2 Introduction

- Class is about *designing* and *analyzing algorithms*
  - *Algorithm*: A well-defined procedure that transfers an input to an output.
    * Not a program (but often specified like it): An algorithm can often be implemented in several ways.
  - *Design*: We will study methods/ideas/tricks for developing (fast!) algorithms.
  - *Analysis*: Abstract/mathematical comparison of algorithms (without actually implementing them).
- Math is needed in three ways:
  - Formal specification of problem
  - Analysis of correctness
  - Analysis of efficiency (time, memory use,...)
- Hopefully the class will show that **algorithms matter!**

3 Algorithm example: Insertion-sort

3.1 Specification

- Input: n integers in array $A[1..n]$
- Output: A sorted in increasing order

3.2 Insertion-sort algorithm

```
FOR j = 2 to n DO
   key = A[j]
   i = j − 1
   WHILE i > 0 and A[i] > key DO
      A[i + 1] = A[i]
      i = i − 1
   OD
   A[i + 1] = key
OD
```

- NOTE: Algorithm shows example of the (Pascal like) pseudo-code we will sometimes used to describe algorithms.

Example:

5 2 4 6 1 3 j=2 i=1 key=2

5 5 4 6 1 3 i=0

2 5 | 4 6 1 3

2 5 4 6 1 3 j=3 i=2 key=4

2 5 5 6 1 3 i=1

2 4 5 | 6 1 3

2 4 5 6 1 3 j=4 i=3 key=6

2 4 5 6 | 1 3

2 4 5 6 1 3 j=5 i=4 key=1

2 4 5 6 6 3 i=3

2 4 5 5 6 3 i=2

2 4 4 5 6 3 i=1

2 2 4 5 6 3 i=0

1 2 4 5 6 | 3

1 2 4 5 6 3 j=6 i=5 key=3

1 2 4 5 6 6 i=4

1 2 4 5 5 6 i=3

1 2 4 4 5 6 i=2

1 2 3 4 5 6 |

3.3 Correctness

- Induction:
  - The *Invariant* "A[1..j-1] is sorted" holds at the beginning of each iteration of FOR-loop.
  - When j=n+1 we have correct output.

3.4 Analysis

- We want to predict the resource use of the algorithm.
- We can be interested in different resources
  - but normally *running time*.
- To analyze running time we need mathematical model of a computer:

Random-access machine (RAM) model:

- Memory consists of infinite array
- Instructions executed sequentially one at a time
- All instructions take unit time:
  * Load/Store
  * Arithmetics (e.g. $+, -, *, /$)
  * Logic (e.g. $>$)

- Running time of an algorithm is the number of RAM instructions it executes.
- RAM model not completely realistic, e.g.
  - memory not infinite (even though we often imagine it is when we program)
  - not all memory accesses take same time (cache, main memory, disk)
  - not all arithmetic operations take same time (e.g. multiplications expensive)
  - instruction pipelining
  - other processes
- But RAM model often enough to give relatively realistic results (if we don't cheat too much).

- Running time of insertion-sort depends on many things
  - How sorted the input is
  - How big the input it
  - ...
- Normally we are interested in running time as a function of *input size*
  - in insertion-sort: n.
- We don't really want to count every RAM instruction
  - Let us analyze insertion-sort by assuming that line i in the program use c_i RAM instructions.
  - How many times are each line of the program executed?
    * Let t_j be the number of times line 4 (the WHILE statement) is executed in the j'th iteration.

	cost	times
FOR $j = 2$ to n DO	c_1	n
$key = A[j]$	c_2	$n-1$
$i = j - 1$	c_3	$n-1$
WHILE $i > 0$ and $A[i] > key$ DO	c_4	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_5	$\sum_{j=2}^{n}(t_j - 1)$
$i = i - 1$	c_6	$\sum_{j=2}^{n}(t_j - 1)$
OD		
$A[i+1] = key$	c_7	$n-1$
OD		

- Running time: (depends on t_j)

  $T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4\sum_{j=2}^{n} t_j + c_5\sum_{j=2}^{n}(t_j - 1) + c_6\sum_{j=2}^{n}(t_j - 1) + c_7(n-1)$

  - Best case: $t_j = 1$ (already sorted)

    $$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_3(n-1) + c_4(n-1) + c_7(n-1) \\ &= (c_1 + c_2 + c_3 + c_4 + c_7)n - (c_2 + c_3 + c_4 + c_7) \\ &= k_1 n - k_2 \end{aligned}$$

    Linear function of n

  - Worst case: $t_j = j$ (sorted in decreasing order)

    $$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_3(n-1) + c_4\sum_{j=2}^{n} j + c_5\sum_{j=2}^{n}(j-1) + c_6\sum_{j=2}^{n}(j-1) + c_7(n-1) \\ &= c_1 n + c_2(n-1) + c_3(n-1) + c_4(\tfrac{n(n+1)}{2} - 1) + c_5(\tfrac{(n-1)n}{2}) + c_6(\tfrac{(n-1)n}{2}) + c_7(n-1) \\ &= (c_4/2 + c_5/2 + c_6/2)n^2 + (c_1 + c_2 + c_3 + c_4/2 - c_5/2 - c_6/2 + c_7)n - (c_2 + c_3 + c_4 + c_7) \\ &= k_3 n^2 + k_4 n - k_5 \end{aligned}$$

    Quadratic function of n

    Note: We used $\boxed{\sum_{j=1}^{n} j = \frac{n(n+1)}{2}}$ (Next week!)

  - "Average case": Be careful! (average over what?)

    We assume n numbers chosen randomly $\Rightarrow t_j = j/2$

    $T(n) = k_6 n^2 + k_7 n + k_8$

    Still **Quadratic function of n**

- Note:

  - We will normally be interested in worst-case running time.
    - Upper bound on running time for *any* input.
    - For some algorithms, worst-case occur fairly often.
    - Average case often as bad as worst case (but not always!).
  - We will only consider order of growth of running time:
    - We already ignored cost of each statement and used the constants c_i.
    - We even ignored c_i and used k_i.
    - We just said that best case was *linear in n* and worst/average case *quadratic in n*.

    $\Rightarrow$ O-notation (best case $O(n)$, worst/average case $O(n^2)$) (next lecture!)

4 Designing Good Algorithms: Divide and Conquer/Mergesort

4.1 Divide-and-conquer

- Can we design better than $O(n^2)$ sorting algorithm?
- We will do so using one of the most powerful algorithm design techniques.

Divide and Conquer

To Solve P:

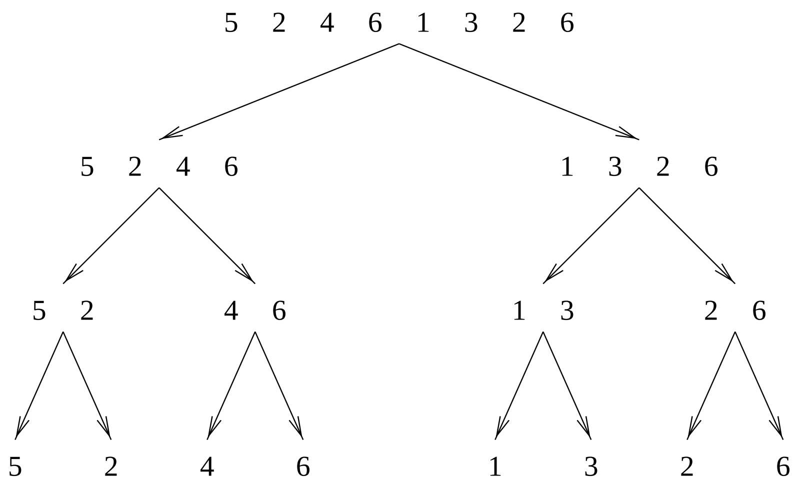
1. *Divide* P into smaller problems $P_1, P_2, P_3 .....P_k$.
2. *Conquer* by solving the (smaller) subproblems recursively.
3. *Combine* solutions to $P_1, P_2, ...P_k$ into solution for P.

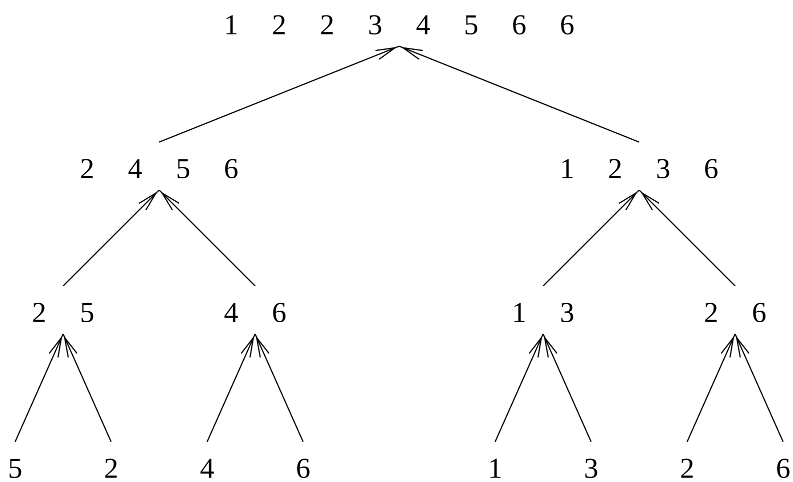
4.2 Merge-Sort

- Using divide-and-conquer, we can obtain a merge-sort algorithm.
  - Divide: Divide n elements into two subsequences of $n/2$ elements each.
  - Conquer: Sort the two subsequences recursively.
  - Combine: Merge the two sorted subsequences.
- Assume we have procedure Merge(A, p, q, r) which merges sorted A[p..q] with sorted A[q+1....r] in $O(r - p)$ time.
- We can sort A[p...r] as follows (initially p=1 and r=n):

```
Merge Sort(A,p,r)
    If p < r then
        q = ⌊(p + r)/2⌋
        MergeSort(A,p,q)
        MergeSort(A,q+1,r)
        Merge(A,p,q,r)
```

Example:





4.3 Correctness

- Induction on n
  - Easy assuming Merge() is correct!

4.4 Analysis

- To simplify things, let us assume that n is a power of 2, i.e $n = 2^k$ for some k.
- Running time of the procedure can be analyzed using a recurrence equation/relation.

$$\begin{aligned} T(n) &\leq c_1 + T(n/2) + T(n/2) + c_2 n \\ &\leq 2T(n/2) + c_3 n \end{aligned}$$

$\Downarrow$

$T(n) = O(n \log_2 n)$ as we will see later.

- We can also get $O(n \log_2 n)$ bound by noticing that the recursion tree has depth $\log_2 n$ and that $O(n)$ time is spent on each level.

- Note:
  - We really have $T(n) = c_4$ if $n = 1$
  - If $n \neq 2^k$ things can be complicated (We will often assume $n = 2^k$ to avoid complicated cases).

4.5 Log's

- Base 2 logarithm comes up all the time (from now on we will always mean $\log_2 n$ when we write $\log n$).
  - Number of times we can divide n by 2 to get to 1 or less.
  - Number of bits in binary representation of n.
  - Inverse function of $2^n = 2 \cdot 2 \cdot 2 \cdots 2$ (n times).
  - Way of doing multiplication by addition: $\log(ab) = \log(a) + \log(b)$
- Note:
  - $\log_a n = \frac{\log_b n}{\log_b a}$
  - $\log n << \sqrt{n} << n$