Topic 14: Splay Trees

[Kozen, 12]

CPS 230, Fall 2001

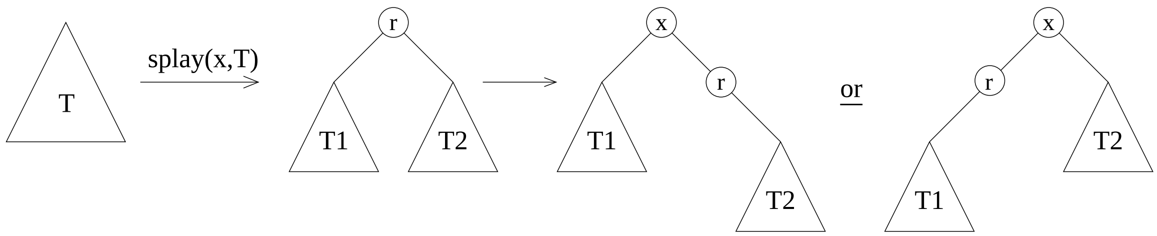
1 Splay trees

- We have previously discussed binary search trees and how they can be kept balanced ($O(\log n)$ height) during insert and delete operations (red-black trees).
  - Rebalancing rather complicated
  - Extra space used for the color of each node
- We also discussed skip lists which are a lot simpler than red-black trees
  - Only guarantee $O(\log n)$ *expected* performance
  - No extra information is used for rebalance information though
- Splay trees are search trees that "magically" balance themselves (no rebalance information is stored) and have *amortized* $O(\log n)$ performance.
- Recall the basic properties of search trees:
  - Binary tree with elements stored in nodes
  - If node v holds element (key value) e then
    - all elements in left subtree are $< e$
    - all elements in left subtree are $> e$
- Splay tree:
  - Normal (possibly unbalanced) search tree T
  - All operations implemented using one basic operation, SPLAY:

SPLAY(x, T) searches for x in the tree T and reorganizes the tree so that the new root is either x (if x is in T) or else the minimum element $> x$ or the maximum element $< x$ (if x is not in T).

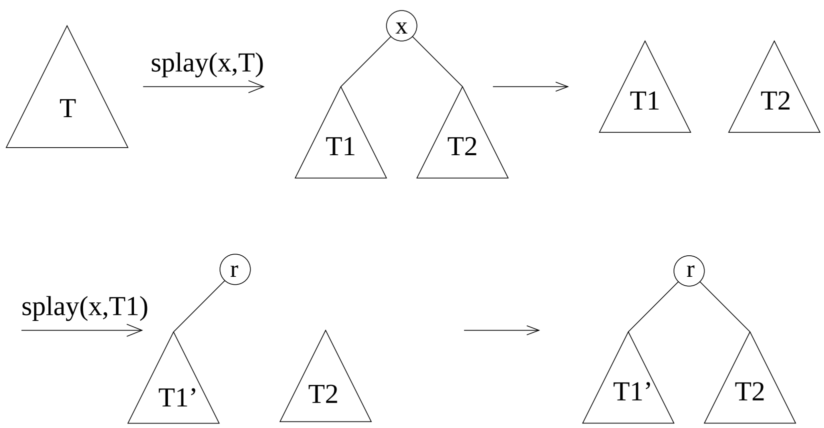
– SEARCH(x, T): SPLAY(x, T) and then inspect the root.

– INSERT(x, T): SPLAY(x, T) and create a new root with x.



– DELETE(x, T):

  * SPLAY(x, T) and remove the root, thus yielding two subtrees $T1$ and $T2$.
  * SPLAY$(x, T1)$.
  * Make $T2$ right son of new root of $T1$ after the SPLAY.



$\Longrightarrow$ All operations perform $O(1)$ SPLAYs and use $O(1)$ extra time.

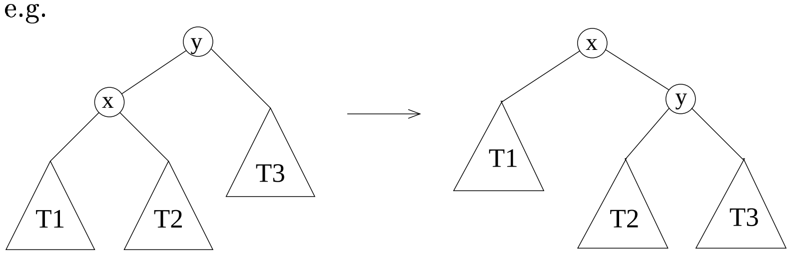
$\Longrightarrow$ If SPLAY runs in $O(\log n)$ amortized time, then so do all operations.

- Implementation of SPLAY:

  – Search for x like in normal search tree

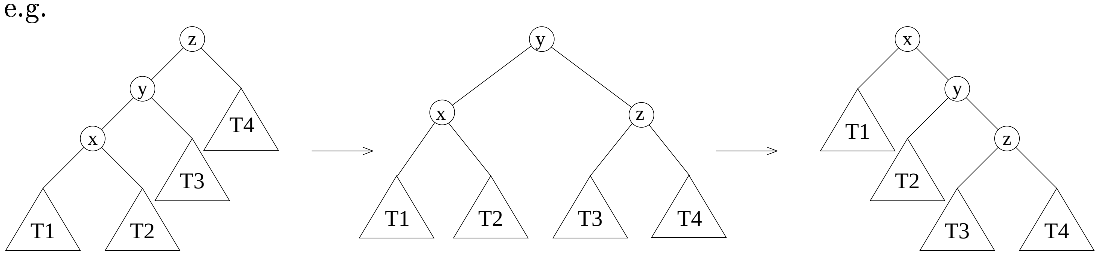
  – Repeatedly rotate x up until it becomes the root.
  We distinguish between three cases:

  1. x is child of root (no grandparent): **Do rotate(x)**
  e.g.

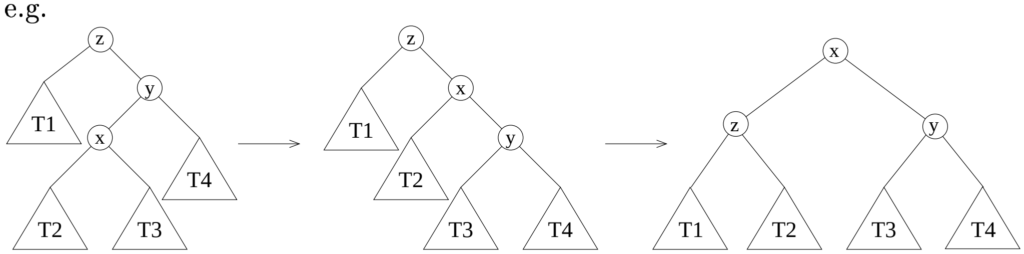


  2. x has parent y *and* grandparent z *and* x and y are both left children or both right

children: **Do rotate**(y) **followed by rotate**(x)
e.g.

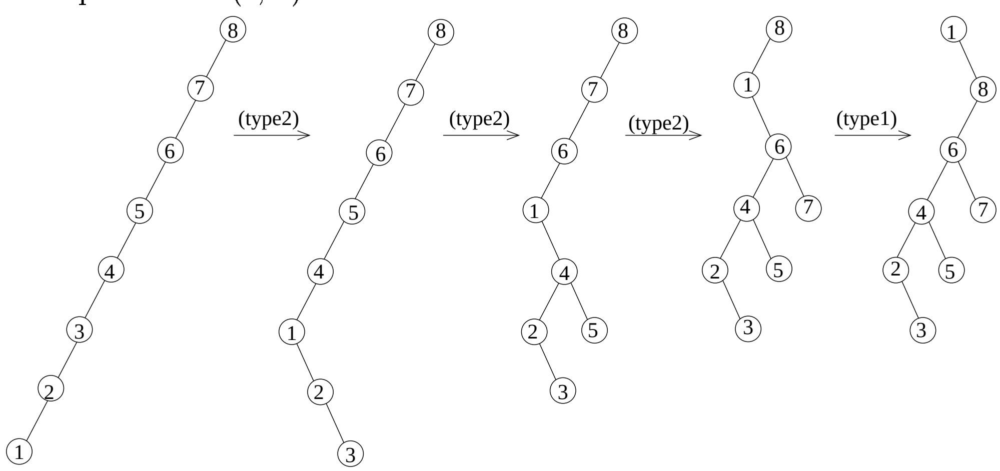


3. x has parent y *and* grandparent z *and* one of x and y is a left child and the other is a right child: **Do rotate**(x) **followed by rotate**(x)
e.g.

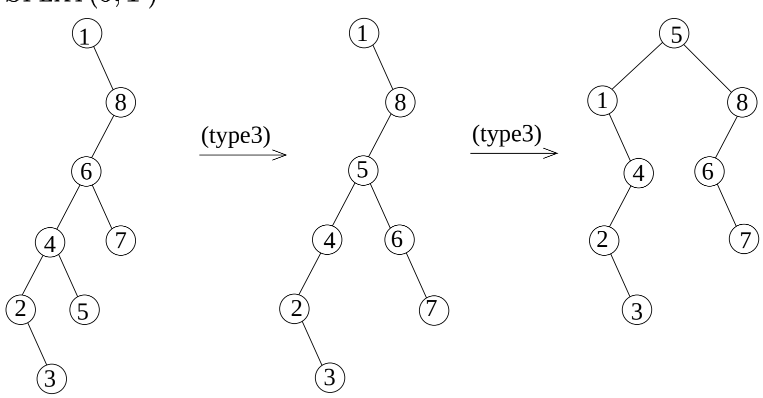


- Note:
  - A Splay can take $O(n)$ worst-case time (very unbalanced tree)
  - But splay trees somehow seem to stay nicely balanced $\Longrightarrow O(\log n)$ amortized Splay.

Examples: Splay$(1, T)$



Splay$(5, T)$



2 Analysis of Splay Trees

- We will use the *accounting method* to show that all operations take $O(\log n)$ amortized time.
  - We will imagine that each node in the tree has credits on it.
  - We will use some credits to pay for (part of) rotations during a SPLAY.
  - We will see that in addition to the SPLAY cost, each INSERT and DELETE requires placing $O(\log n)$ new credits on the new root node.
- We will ignore cost of searching for x, since the rotations cost at least as much as the search. (That is, if we can bound amortized rotation cost, we can also bound search cost.)
- Let $T(x)$ be the subtree of T that is rooted at x. We will maintain the *credit invariant* that the number of credits on each node is

$$\mu(x) = \lfloor \log |T(x)| \rfloor.$$

Equivalently, we could use the potential function

$$\Phi(T) = \sum_x \mu(x).$$

- We will prove the following lemma:

At most $3(\mu(T) - \mu(x) + O(1))$ new credits are needed to perform the SPLAY(x, T) operation and maintain the credit invariant.

- This lemma implies that a SPLAY operation uses at most $3\lfloor \log n \rfloor + O(1) = O(\log n)$ new credits (amortized time).
- In addition, each INSERT or a DELETE requires us to place onto the new root at most $O(\log n)$ new credits, so the total number of new credits placed counting those done by the SPLAY is $O(\log n)$, thus giving the $O(\log n)$ amortized time bound.
- Proof of lemma:
  - Let μ and μ' denote the value of μ before and after a rotate operation.
  - During a SPLAY operation we perform some number (say, k) of case 2 and 3 operations and possibly one case 1 operation.
  - We will show that the actual cost of an operation is as follows:
    * Case 1: $3(\mu'(x) - \mu(x)) + O(1)$
    * Case 2: $3(\mu'(x) - \mu(x))$
    * Case 3: $3(\mu'(x) - \mu(x))$

  $\Longrightarrow$ When we sum the actual costs over all $\leq k+1$ operations in the SPLAY, we get

$$3(\mu(T) - \mu(x)) + O(1), \tag{1}$$

  where $\mu(x)$ is the number of credits on x before the SPLAY.
  - Note that it is important that the additive $O(1)$ term appears only in case 1. If the $O(1)$ additive term also appeared in cases 2 and 3, we would get an additive $O(k)$ term in (1), so this approach wouldn't work.

- Case 1:
  - We have $\mu'(x) = \mu(y)$, $\mu'(y) \leq \mu'(x)$, and all other μ's are unchanged.
  - To maintain invariant, we use the following number of credits:

$$\begin{aligned} \mu'(x) + \mu'(y) - \mu(x) - \mu(y) &= \mu'(y) - \mu(x) \\ &\leq \mu'(x) - \mu(x) \\ &\leq 3(\mu'(x) - \mu(x)) \end{aligned}$$

  - To do the actual rotation, we use $O(1)$ credits.
- Case 2:
  - We have $\mu'(x) = \mu(z)$, $\mu'(y) \leq \mu'(x)$, $\mu'(z) \leq \mu'(x)$, $\mu(y) \geq \mu(x)$, and all other μ's are unchanged.
  - To maintain the invariant, we use the following number of credits:

$$\begin{aligned} \mu'(x) + \mu'(y) + \mu'(z) - \mu(x) - \mu(y) - \mu(z) &= \mu'(y) + \mu'(z) - \mu(x) - \mu(y) \\ &= (\mu'(y) - \mu(x)) + (\mu'(z) - \mu(y)) \\ &\leq (\mu'(x) - \mu(x)) + (\mu'(x) - \mu(x)) \\ &= 2(\mu'(x) - \mu(x)) \end{aligned}$$

  - This means that we can use the remaining $\mu'(x) - \mu(x)$ credits to pay for rotation, *unless* $\mu'(x) = \mu(x)$ (which can happen because of the floor function, since $\mu(x) = \lfloor \log |T(x)| \rfloor$).
  - We will show by contradiction that if $\mu'(x) = \mu(x)$ then $\mu'(x) + \mu'(y) + \mu'(z) < \mu(x) + \mu(y) + \mu(z)$, which means that the operation actually *releases* credits, which we can use for the rotation:
    * Assume that $\mu'(x) = \mu(x)$. To set up the proof by contradiction, let's assume that $\mu'(x) + \mu'(y) + \mu'(z) \geq \mu(x) + \mu(y) + \mu(z)$
    * We have $\mu(z) = \mu'(x) = \mu(x)$ and $\mu(x) \leq \mu(y) \leq \mu(z)$
      $\Longrightarrow \quad \mu(z) = \mu(x) = \mu(y)$
      $$\Longrightarrow \quad \begin{aligned} \mu'(x) + \mu'(y) + \mu'(z) &\geq \mu(x) + \mu(y) + \mu(z) \\ &= 3\mu(x) \\ &= 3\mu'(x) \end{aligned}$$
      $\Longrightarrow \quad \mu'(y) + \mu'(z) \geq 2\mu'(x)$.
    * Since $\mu'(y) \leq \mu'(x)$ and $\mu'(z) \leq \mu'(x)$, we get $\mu'(x) = \mu'(y) = \mu'(z)$.
    * Therefore, we have

$$\mu(x) = \mu(y) = \mu(z) = \mu'(x) = \mu'(y) = \mu'(z) \tag{2}$$

    * We will now show that (2) cannot possibly be true (which will complete the proof by contradiction):
      Let a be $|T(x)|$ before the rotations (i.e., $a = |T1| + |T2| + 1$).
      Let b be $|T(z)|$ after rotations (i.e., $b = |T3| + |T4| + 1$).
      Since $\mu(x) = \mu'(z) = \mu'(x)$, we have $\lfloor \log a \rfloor = \lfloor \log b \rfloor = \lfloor \log(a + b + 1) \rfloor$, but then we have the following contradiction:
      · if $a \leq b$, then $\lfloor \log(a + b + 1) \rfloor \geq \lfloor \log 2a \rfloor = 1 + \lfloor \log a \rfloor > \lfloor \log a \rfloor$
      · if $a > b$, then $\lfloor \log(a + b + 1) \rfloor \geq \lfloor \log 2b \rfloor = 1 + \lfloor \log b \rfloor > \lfloor \log b \rfloor$
- Case 3:
  - Can be proved analogously to case 2.