Algorithms
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ALG 1.2
Asymptotics and Recurrence Equations

(a) **Computational Complexity of a Program**
(b) **Worst Case and Expected Bounds**
(c) **Solution of Recurrence Notation**
(d) **Definition of Asymptotic Equations**

Main Reading Selections:
CLR, Chapter 2, 3, 4
Handout: "Counting and Estimating

Auxillary Reading Selections:
AHU-Design, Chapter 2
AHU-Data, Chapter 9
BB-Chapter 2

Asymptotics:

goal
is to estimate and compare growth rates of functions

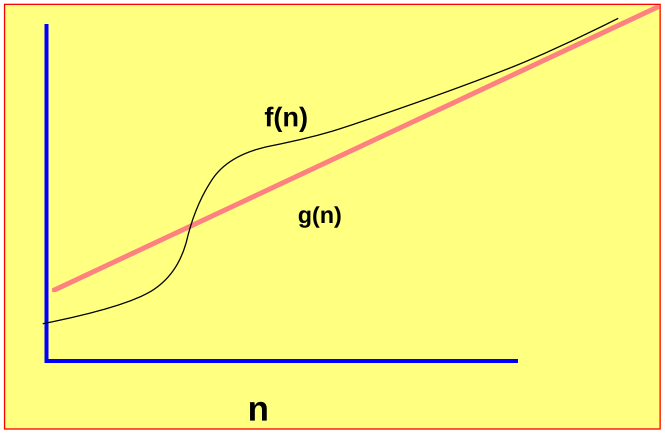
ignore
constant factors of
growth

"f(n) is *asymptotically equal* to g(n)"

$$f(n) \sim g(n)$$

if

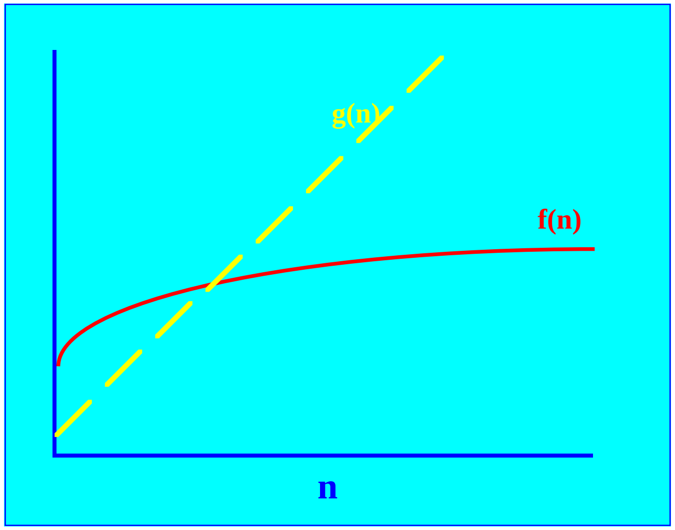
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$



"f(n) is *little-o* g(n)"

f(n) is o(g(n)) if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$



"f(n) is *big - O* g(n)"

f(n) is O(g(n)) if

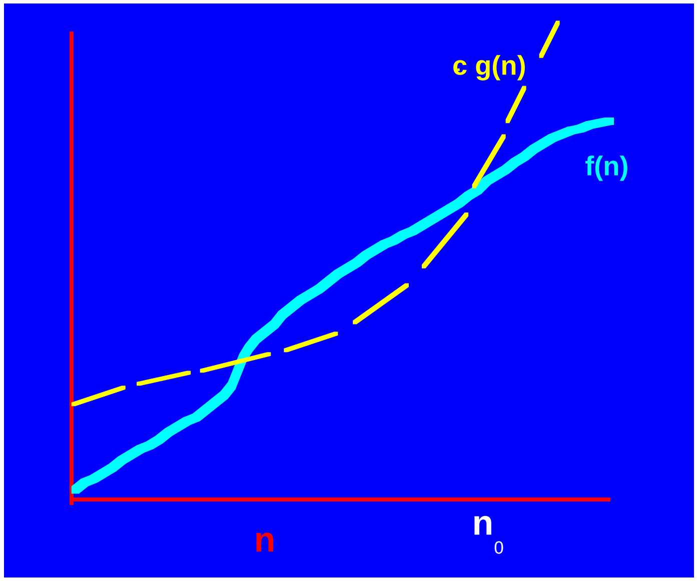
$$\limsup_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| \le c$$

ie ,

$\exists$ c, n_0 > 0

s.t. $f(n) \le c \cdot g(n)$

for all $n \ge n_0$

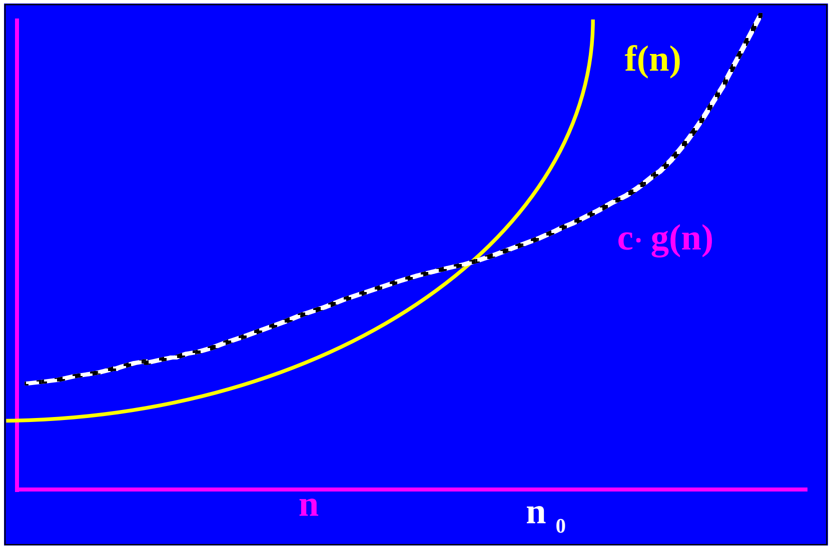


"f(n) is *order at least* g(n)"

f(n) is Ω (g(n)) if

$$\liminf_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| \geq c$$

$\exists\ n_0, c > 0$ s.t. $f(n) \geq c\, g(n)$ for all $n \geq n_0$

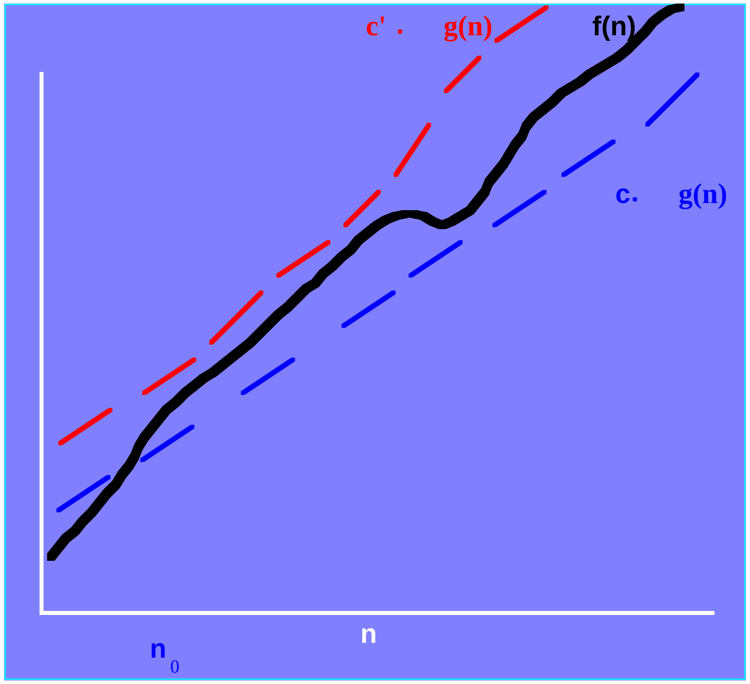


"f(n) is *order tight* with g(n)
f(n) is θ(g(n)) if

f(n) is O(g(n)) and also Ω(g(n))

ie

$\exists\, n_0, c, c'$ s.t. $c \cdot g(n) \le f(n) \le c' \cdot g(n)$



Suppose my algorithm
runs in time O(n)

don't say: "his runs in time O(n^2)
so is worse"

but prove: "his runs in time Ω(n^2)
so is worse"

- must find a *worst case input* of length n for which his algorithm takes time $\ge cn^2$ for all $n \ge n_0$

Notation

n is $O(n^2)$
sometimes written
$n = O(n^2)$

but n^2
is *not*
$O(n)$
so *can't*
use identities!

⇒ The two sides of the equality *do not* play a symmetric role

write "$f(n) - g(n)$ is $o(h(n))$"

as $f(n) = g(n) + o(h(n))$

example

$$\frac{n}{n-1} = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$
$$= 1 + \frac{1}{n} + o\left(\frac{1}{n}\right)$$
$$= 1 + o(1) \quad \text{as } n \to \infty$$

Convergent Power Sum

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \leq O(1)$$

for $0 < x < 1$

A polynomial is

asymptotically

equal to its leading term

as $x \to \infty$

$$\sum_{i=0}^{d} a_i x^i = \theta\left(x^d\right)$$

$$\sum_{i=0}^{d} a_i x^i = o\left(x^{d+1}\right)$$

$$\sum_{i=0}^{d} a_i x^i \sim a_d x^d$$

Sums of Powers:
for $n \to \infty$

$$\sum_{i=1}^{n} i^d \sim \frac{1}{d+1} n^{d+1}$$

or equivalently

$$\sum_{i=1}^{n} i^d = \frac{1}{d+1} n^{d+1} + o\left(n^{d+1}\right)$$

examples

$$\sum_{i=1}^{n} i \sim \frac{n^2}{2}$$

$$\sum_{i=1}^{n} i^2 \sim \frac{n^3}{3}$$

2nd order asymptotic expansion

$$\sum_{i=1}^{n} i^d = \frac{1}{d+1} n^{d+1} + \frac{1}{2} n^d + O\left(n^{d-1}\right)$$

Asymptotic Expansion of f(n) as $n \to n_0$

$$f(n) \sim \sum_{i=1}^{\infty} c_i \, g_i(n)$$

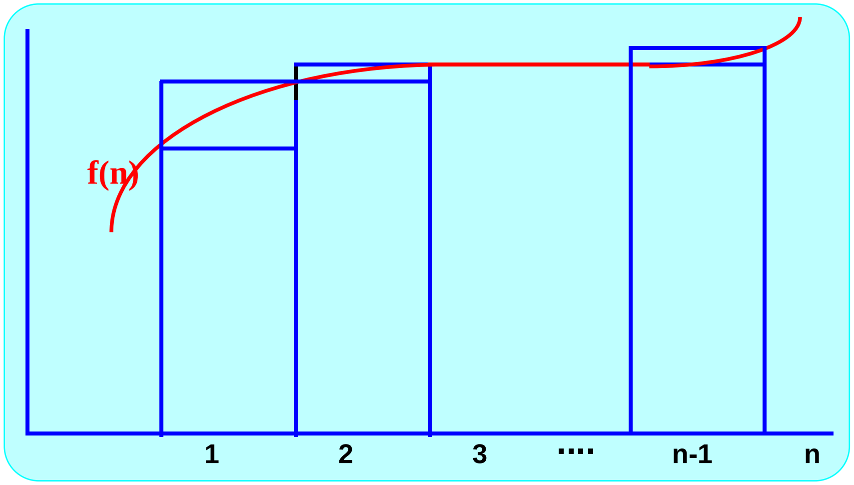
if

(1) $g_{i+1}(n) = o(g_i(n))$ for all $i \geq 1$

and

(2) $f(n) = \sum_{i=1}^{k} c_i \, g_i(n) + o(g_k(n))$ for all $k \geq 1$

Bounding Sums by Integrals



$$\sum_{k=1}^{n} f(k) \le \int_{1}^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k+1)$$

so $\int_{1}^{n+1} f(x)dx - f(n+1) + f(1) \le \sum_{k=1}^{n} f(k) \le \int_{1}^{n+1} f(x)dx$

example if $f(x) = \ln(x)$ then $\int \ln(x)dx = x\ln(x) - x$

so $\sum_{k=1}^{n} \ln k = (n+1)\ln(n+1) - n + \theta(\ln(n))$

since $\log(n) = \frac{\ln n}{\ln 2}$

so $\sum_{k=1}^{n} \log k = (n+1)\log(n+1) - \frac{n}{\ln 2} + \theta(\log n)$

Other Approximations Derived from Integrals

$$\sum_{k=1}^{n} k \log k = \frac{(n+1)^2}{2}\log(n+1) - \frac{(n+1)^2}{4\ln 2} + \theta(n \log n)$$

Harmonic Numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right)$$

Euler's constant $\gamma = .577\ldots$

factorial $n! = 1 \cdot 2 \cdot 3 \cdots n$

Stirling's Approximation for Factorial

$$n! \sim \sqrt{2\pi n}\; n^{n}\; e^{-n} \text{ as } n \to \infty$$

So,

$$\log(n!) = n\log n - n\log e + \tfrac{1}{2}\log(2\pi n) + \theta(1)$$
$$= n\log n - \theta(n)$$

Recurrence Equations
(over integers)

homogeneous of degree d

$n > d$

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_d x_{n-d}$$

given (constant coefficients $a_1, \ldots, a_d$
initial values $x_1, x_2, \ldots, x_d$

example:

Fibonacci Sequence

n≥2

$$F_n = F_{n-1} + F_{n-2}$$

$$F_0 = 0, \; F_1 = 1$$

Solution of Fibonacci Sequence

$$r_1 = \frac{1}{2}(1 + \sqrt{5}) = 1.618 \ldots \text{ "golden ratio"}$$

$$r_2 = \frac{1}{2}(1 - \sqrt{5})$$

$$F_n = c_1 r_1^n + c_2 r_2^n$$

where $F_0 = c_1 + c_2 = 0$

$$F_1 = c_1 r_1 + c_2 r_2 = 1$$

1

hence $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$

$$\Rightarrow F_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$$

Homogeneous Recurrence Relations

(no constant additive term)

Solve: $x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_d x_{n-d}$

try $x_n = r^n$

multiply by r^d / r^n

get characteristic equation:

$$r^d - a_1 r^{d-1} - a_2 r^{d-2} - \dots - a_d = 0$$

Case

Distinct Roots

$r_1, r_2, \dots, r_d$

$$\Rightarrow x_n = \sum_{i=1}^{d} c_i r_i^n$$

$$x_n \sim c_i r_i^n$$

where r_i is **dominant root**

$|r_i| > |r_j| \quad \forall\ j \neq i$

Case

Roots are not distinct

$r_1 = r_2 = r_3$

Then solutions not independent, so additional terms:

$$x_n = c_1 r_1^n + c_2 n r_1^n + c_3 n^2 r_1^n + \sum_{i=4}^{d} c_i r_i^n$$

Inhomogeneous Recurrence Equation

$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_d x_{n-d} + a_0$

nonzero constant term $a_0 \neq 0$

Solution Method

(1) Solve homogeneous equation

$Y_n = a_1 Y_{n-1} + a_2 Y_{n-2} + \ldots + a_n Y_{n-d}$

(2)

Case

$\Sigma a_i \neq 1$, add particular solution

$$x_n = c = \frac{a_0}{1 - \Sigma a_i}$$

Case

$\Sigma a_i = 1$, add particular solution

$$x_n = cn = \left(\frac{a_0}{\Sigma i a_i}\right) n$$

(3)

add particular and homogeneous solutions, and solve for constants

This is all we usually need!!

A Useful *Theorem*

$c>0 \quad d>0,$

if $T(n) = \begin{cases} c_0 & n=1 \\ aT\left(\frac{n}{b}\right) + cn^d & n>1 \end{cases}$

then $T(n) = \begin{cases} \theta\left(n^{\log_b a}\right) & a>b^d \\ \theta\left(n^d \log_b n\right) & a=b^d \\ \theta\left(n^d\right) & a<b^d \end{cases}$

Proof

$T(n) = cn^d g(n) + a^{\log_b n} d$ is solution

$g(n) = 1 + \frac{a}{b^d} + \left(\frac{a}{b^d}\right)^2 + \ldots + \left(\frac{a}{b^d}\right)^{\log_b n-1}$

Cases

(1) $a > b^d \Rightarrow g(n) \sim \left(\frac{a}{b^d}\right)^{\log_b n-1}$

is last term so

$T(n) = \theta\left(a^{\log_b n} d\right) = \theta\left(n^{\log_b a}\right)$

(2) $a=b^d \Rightarrow g(n) = \log_b n$

so $T(n) = \theta\left(n^d \log_b n\right)$

(3) $a<b^d \Rightarrow g(n)$ upper bound by $0(1)$

so $T(n) = \theta\left(n^d\right)$

Example

mergesort

input list L of length N

if N=1 *then* return L

else do

let L_1 be the first $\lfloor \frac{N}{2} \rfloor$ elements of L

let L_2 be the last $\lceil \frac{N}{2} \rceil$ elements of L

$M_1 \leftarrow$ Mergesort (L_1)

$M_2 \leftarrow$ Mergesort (L_2)

return Merge (M_1 , M_2)

Time Bound

Initial Value $T(1) = c_1$

for N>1 $T(N) \le T\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + T\left(\left\lceil \frac{N}{2} \right\rceil\right) + c_2 N$

for some constants c_1 , $c_2 \ge 1$

N>1 $T(N) \le 2T\left(\frac{N}{2}\right) + c_2 N$

guess $T(N) \le a N \log N + b$

$$\le 2\left(a \frac{N}{2} \log\left(\frac{N}{2}\right) + b\right) + c_2 N$$

holds if $a = c_1 + c_2$, $b = c_1$

Solution $T(N) \le (c_1 + c_2) N \log N + c_1$

$N>1 \quad T(N) \leq 2\, T\left(\frac{N}{2}\right) + c_2 N, \quad T(1) = c_1$

Transform Variables

$n = \log N, \quad N = 2^n$

$n - 1 = \log N - \log 2 = \log\left(\frac{N}{2}\right)$

recurrence equation:

$X_n = T(2^n) = 2\, X_{n-1} + c_2\, 2^n$

$X_0 = T(2^0) = T(1) = c_1$

Solve by usual methods for recurrence equations

$X_n = O(n 2^n)$

so $T(N) = O(N \log N)$