Algorithms
Professor John Reif

ALG 5.3

Flow Algorithms:

(a) Max-flow, min-cut Theorem
(b) Augmenting Paths
(c) 0 - 1 flow
(d) Vertex Connectivity
(e) Planar Flow

Main Reading Selections:
CLR, Chapter 27

Auxillary Reading Selections:
Handout: "Network Flows"
Combinatorial Optimization, by Eugene Lawler, Holt, Rinehart, Winston, 1976.

Network Definition:

digraph $G = (V, E)$

distinguished vertices:

source $s \in V$

sink $t \in V$

edge capacities: $c: E \rightarrow R^+$

Reverse Edges: $E^R =$ reverse of edges in E

$u \longrightarrow v$ $\quad$ $u \longleftarrow v$

(u,v) $\quad$ $(u,v)^R$ $\quad$ $(u,v)^R = (v,u)$

Flow $f: (E \cup E^R) \rightarrow R^+$

$\forall e \in E$ (1) $f(e) = -f(e^R)$

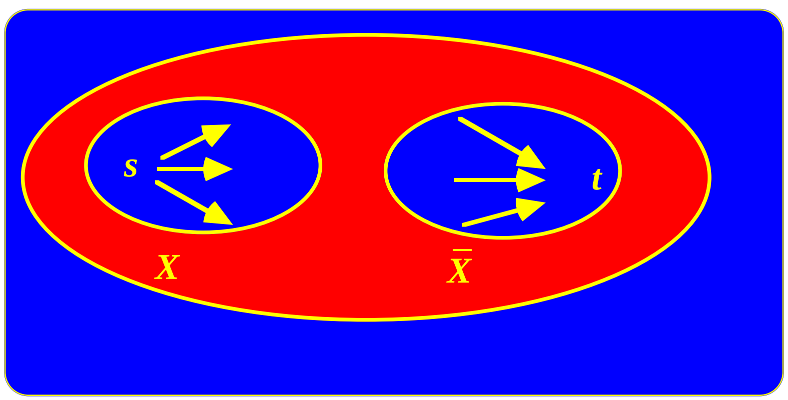
(2) $f(e) \leq c(e)$

$\forall v \in V - \{s,t\}$ (3) $\sum_{(v,u) \in E} f(v,u) = 0$

Flow f

Value $(f) = \sum_{v \in V} f(s,v)$

= sum of flow from source s



cut $X, \bar{X}$ is partition of V

where $s \in X$, $t \in \bar{X}$

$$f(X,\bar{X}) = \sum_{\substack{v \in X \\ u \in \bar{X}}} f(v,u)$$

Lemma: The flow across any cut $X, \bar{X}$ is equal to the *value*(f).

proof:

$$f(X,\bar{X}) = \sum_{\substack{v \in X \\ u \in \bar{X}}} f(u,v) = \sum_{\substack{v \in X \\ w \in \bar{X}}} f(v,w)$$

$$= \sum_{v,w \in X} f(v,w) = value(f) - 0 = value(f).$$

Q.E.D.

residual capacity of edge e:

$res(e) = c(e) - f(e)$

residual graph R: use modified capacities

$c'(e) = res(e)$ for $res(e) > 0$

augmenting path p for flow f is path in R from s to t

$$\underline{res(p)} = \min_{e \in p}(res(e))$$

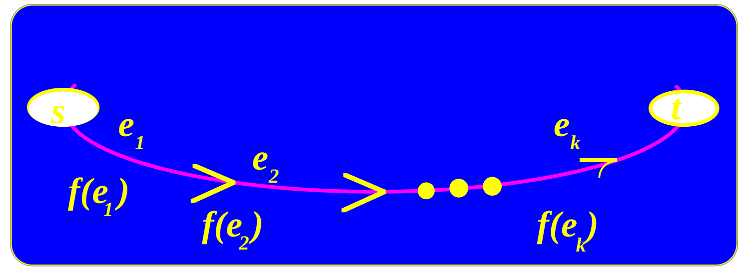
Lemma: R has max flow value $value(f^*) - value(f)$, where f^* is the max flow of G.

proof: If f' is flow in R, then $f + f'$ is flow of G. Also, $f' = f^* - f$ is a flow in R.

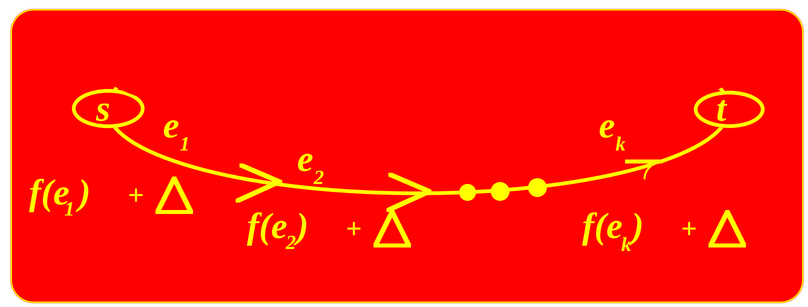
Q.E.D.

Flow f

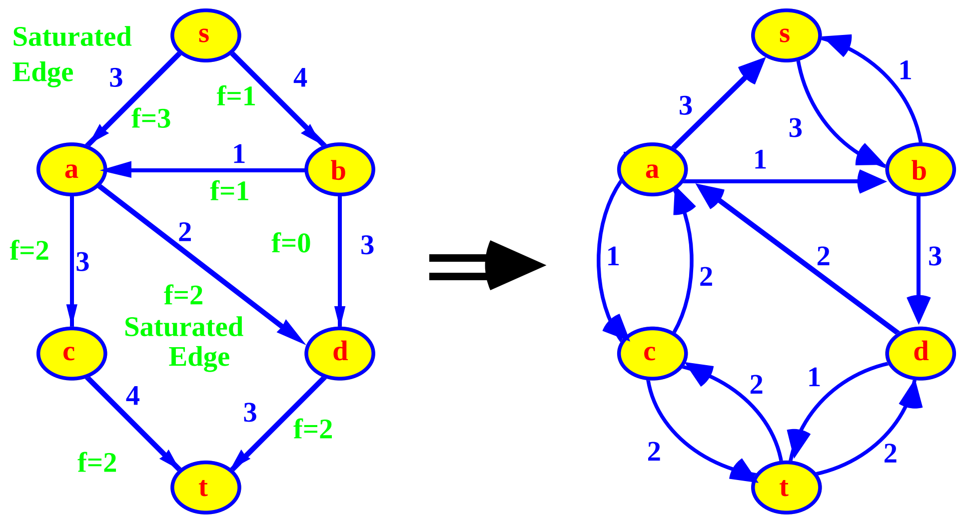
on path $p = (e_1, e_2, \ldots, e_k)$



residual $res(p) = \Delta = \min_{e \in p}(c(e) - f(e))$



gives Augmented flow $f + res(p)$



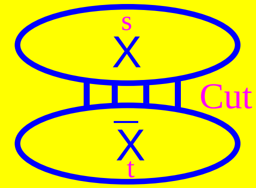
Augmenting Path
(s,b,d,a,c,t)

Network with Flow　　**Residual Network**

<u>min cut</u>: cut of minimum capacity

<u>max flow</u>: $\max(\mathit{value}(f))$

f is flow



<u>Ford - Fulkerson's</u>:

<u>Theorem</u>: The max flow f is equal to the min cut $X, \overline{X}$.

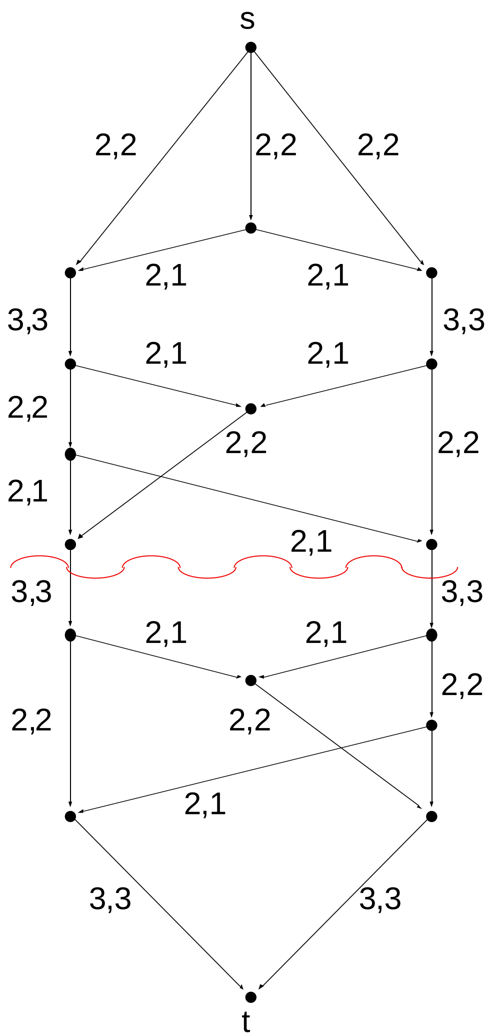
<u>proof</u>: (1) If f is max flow, then there can be no augmenting path from s to t. Let X = vertices in V, reachable from s in residual graph R.

$\overline{X} = V - X$

$$Value(f) = \sum_{\substack{u \in X \\ v \in \overline{X}}} f(u,v) = \sum_{\substack{u \in X \\ v \in \overline{X}}} c(u,v) = c(X, \overline{X})$$

(2) Clearly, f has value at most $c(X, \overline{X})$ for any cut $X, \overline{X}$.

Q.E.D.



Max Flow = Min Cut = 6

Edges Labeled (Capacity, Flow)

Lemma: At most $|E|$ flow augmentations are required to construct max. flow.

proof: Suppose f^* is max flow in G.

Let G^* be subgraph of G with pos. flow.

$i \leftarrow 0$

while $\exists$ path from s to t **do**

- $i \leftarrow i+1$
- find a path p_i from s to t in G^*
- let $\Delta_i = \min_{e \in p_i} f^*(e)$
- **for** all $e \in p$ **do** $f^*(e) \leftarrow f^*(e) - \Delta_i$
  - **if** $f^*(e) \leq 0$, **then** delete e from G^*
  - **od**
- **od**

Note: Deletes at least on edge per step!

Definitions

given flow f

saturated edge e has $f(e) = c(e)$

blocking flow f every path from s to t has saturated edge

(so can not augment flow!)

Idea: Re-route flow

Level Graph L subgraph of R

level(v) = length of shortest path from s to v in R

L contains only edges $(v,u) \in R$ s.t.

level(u) = level$(v) + 1$

Note

L gives shortest augmenting paths

Construct L in $O(|V|+|E|)$ time by Breadth First Search of R

Dinic's Flow Algorithm

Input: network $G = (V,E), s, t$

capacities $c_i : (E \cup E^R) \to R^+$

Initialization: $\forall e,\ f(e) \leftarrow 0$

Loop:

[1] Construct level graph L for f by Breadth First Search.

[2] By augmentations, find blocking flow f' in L from f.

[3] $\forall e,\ f(e) \leftarrow f(e) + f'(e)$

[4] If t is not in level graph,
then return f
else go to [1]

Theorem

Dinic's Algorithm halts after $|V|$ blocking steps.

Proof

Suppose f is current flow with

R = residual graph (currently)

level (v) = min length path from s to v in R

R' = new residual graph

level'(v) = min length of path s to v in R'

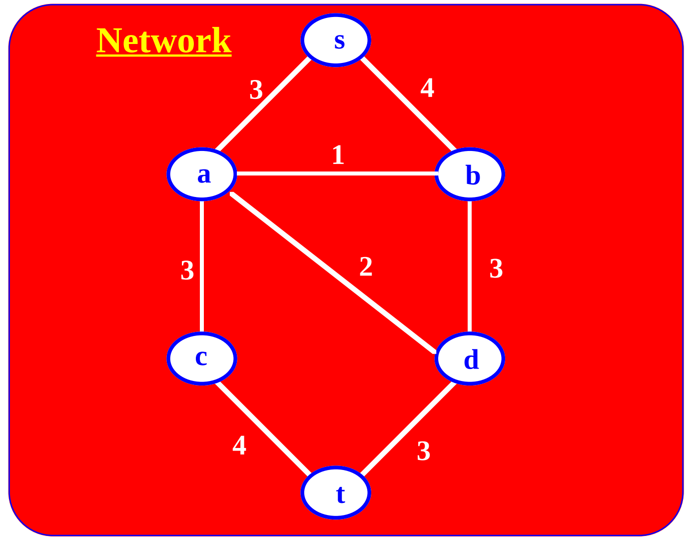
Claim level'(t) > level(t)

Proof (by contradiction)

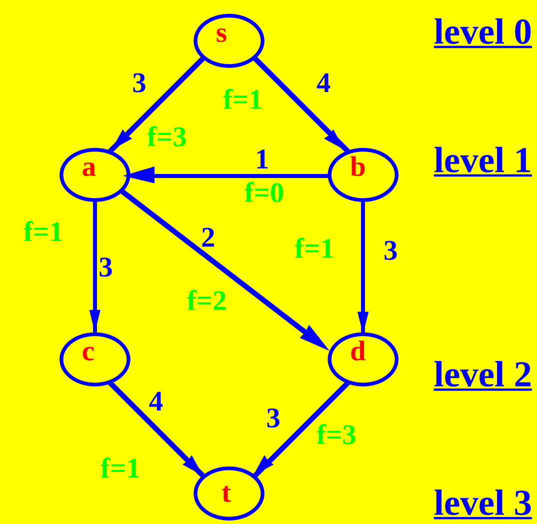
If level(t) = level'(t),

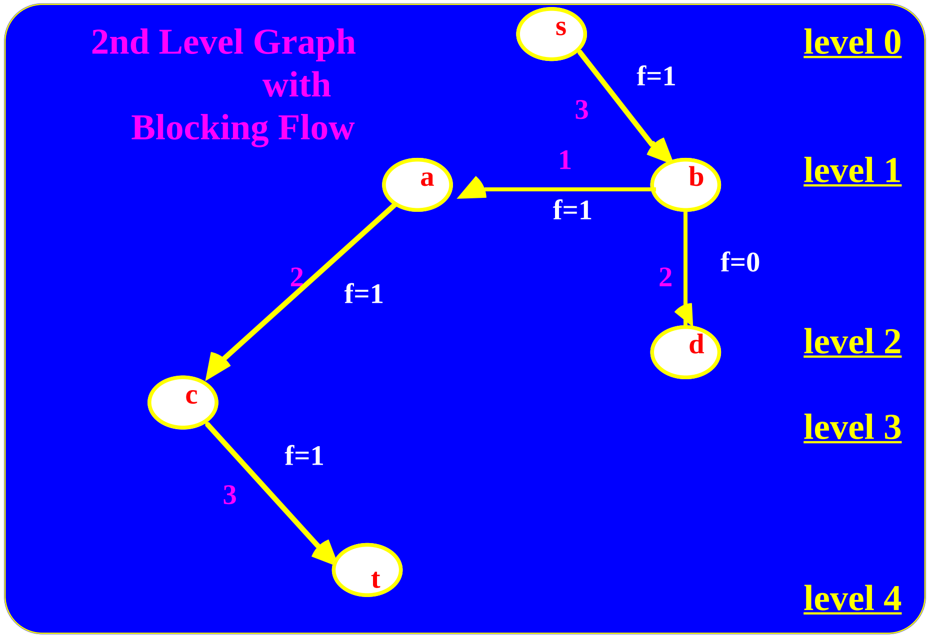
then level(w) = level(v) + 1 for every edge $(v,w) \in L$. This contradicts the fact that $\geq$ one edge is saturated (on the blocking flow) on any path p in L.

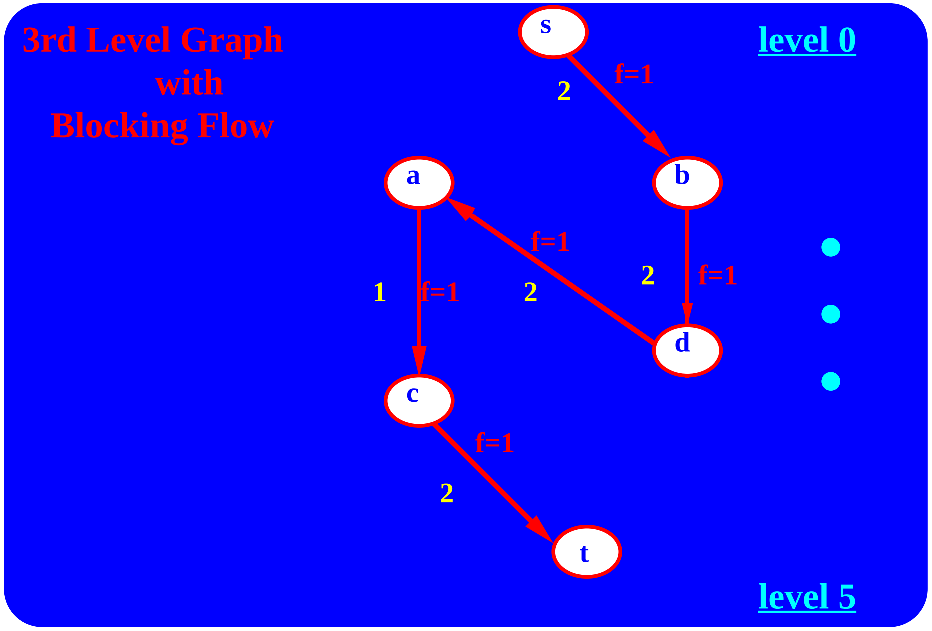
Q.E.D., Hence n steps suffice for the algorithm

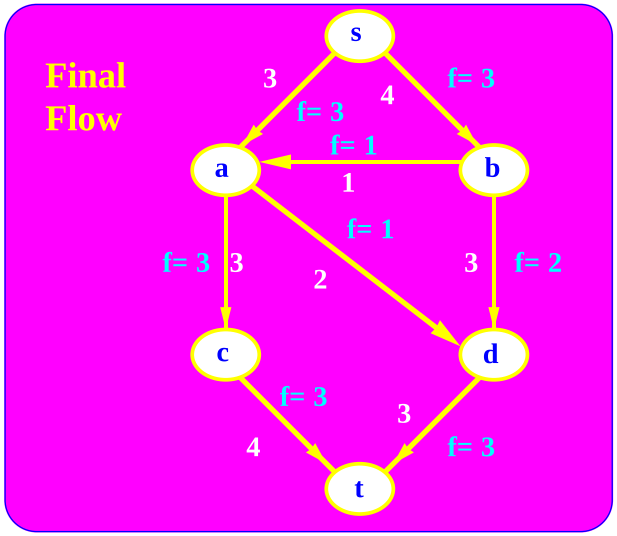


1st Level Graph with Blocking Flow




2nd Level Graph
with
Blocking Flow
s
f=1
3
level 0
a
1
b
level 1
f=1
2
f=1
2
f=0
d
level 2
c
level 3
f=1
3
t
level 4


3rd Level Graph
with
Blocking Flow
s
f=1
2
level 0
a
b
f=1
1
f=1
2
2
f=1
d
c
f=1
2
t
level 5



Finding a Blocking Flow
(Karzanov)

Preflow j:

(1) Satisfies capacities' constraints

(2) May have unbalanced verticies

where $\Delta f(u) = \sum_{u \in v} f(u,v) > 0$

Wave method:

- **begin with blocking preflow f**
  (saturates on edge on every path *s* to *t*
- **balance vertices so $\Delta f(v) = 0$ to get blocking flow**

To balance blocked vertex v:

Repeat (until Δ f (v) =0) do

choose edge (u,v) with f(u,v) > 0

decrease f(u,v) by min (f(uv), Δ f(v))

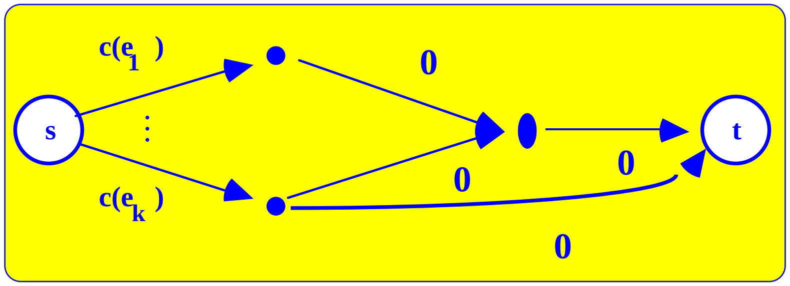
or

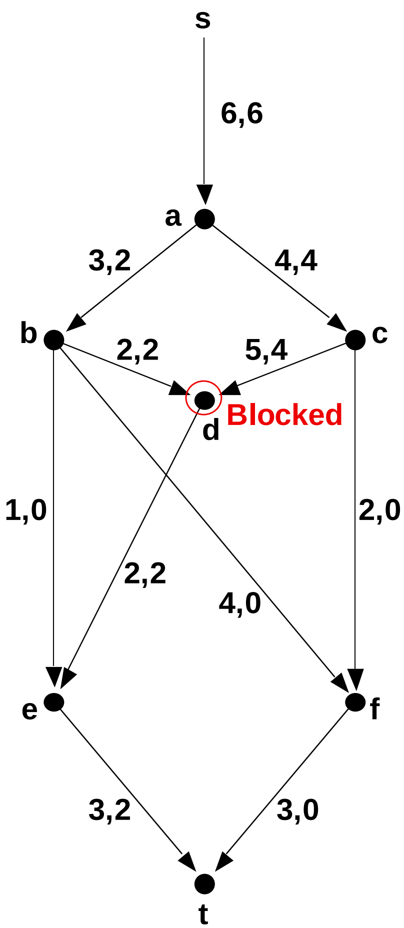
To attempt to balance unblocked vertex v:

Repeat (until $\Delta f(v) = 0$, or there is not an unsaturated edge (v,w) where w is unblocked).
do choose some such edge (v,w) and decrease $f(v,w)$ by $\min\left(C(v,w) - f(v,w), \Delta f(v)\right)$.

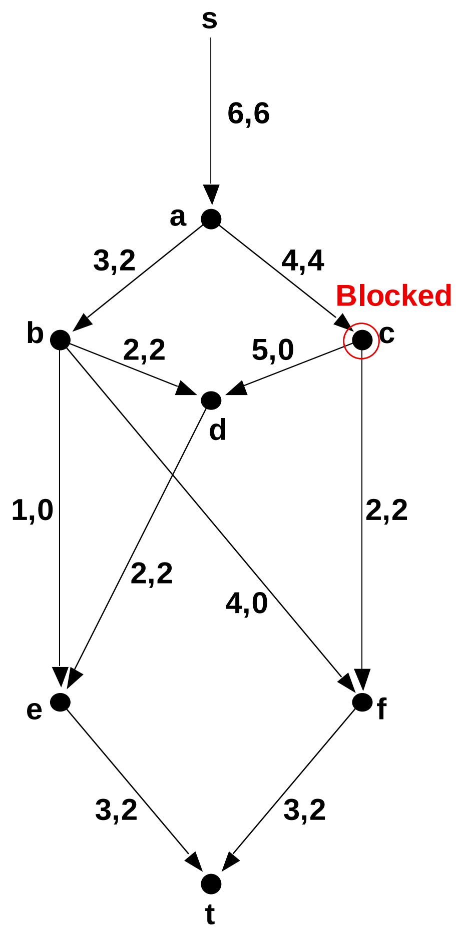
Wave Algorithm for Blocking Flow

Initialize: with preflow that saturates every edge out of *s* and otherwise 0.



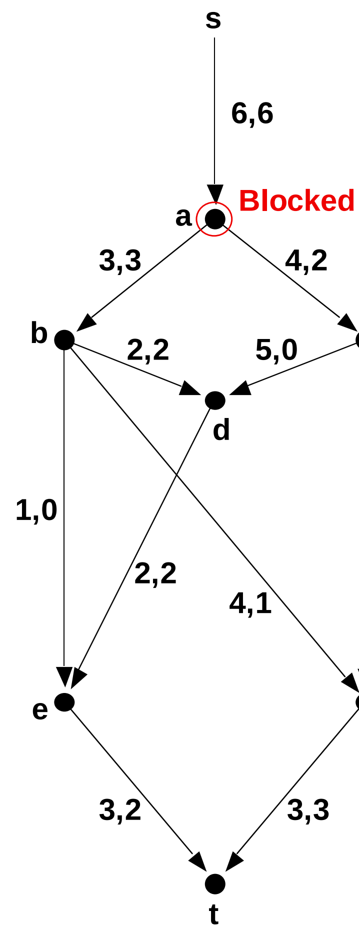


Increased Flow

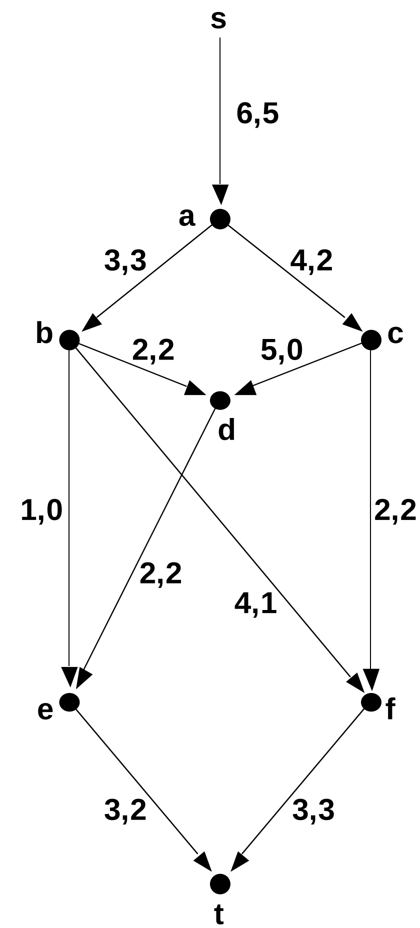


Decreased Flow Balanced *d*

Edges Labeled (Capacity, Flow)



Decreased Flow Balanced *c*



Decreased Flow All Balanced

Edges Labeled (Capacity, Flow)

set s **blocked**, and set $V-\{s\}$ all **unblocked**.

Repeat until there are no unbalanced vertices **do**

Increase flow: Scan all vertices between t, s in topological order, balancing every vertex v that is unbalanced and **unblocked**. (If balancing fails, make v **blocked**.)

Decrease flow: Scan vertices in reverse topological order, balancing each vertex that is unbalanced and **blocked**.

Theorem: Wave Algorithm computes a blocking flow in $O(n^2)$ time (and hence a max flow in $O(n^3)$ time).

Proof (use invariants)**:**

(1) If v blocked $\Longrightarrow$ every path from v to t has saturated edge.

(2) The preflows constructed by algorithm are blocking.

Modify: s blocked, and departing edges saturated.

Inductive Step:

(a) Scanning in topological order in increase flow guarantees no unblocked, unbalanced vertices.

(b) Scanning in reverse topological order guarentees every blocked vertex gets balanced.

Note: Each step blocks at least 1 vertex $\Longrightarrow$ at most n steps flow on edge e increases and decreases at most once $\Longrightarrow$ total time $O(|V|^2+|E|)=O(|V|^2)$

Improved Flow Algorithms

Sleater - Tarjan use data structures to decrease blocking flow algorithms to $O(|E|\log|V|)$ time, giving…

Theorem

Max flow can be computed in $O(|V||E|\log|V|)$ time.

Special Case:

<u>0 − 1 Flow</u>, if $\forall e \in E, c(e) = 1$

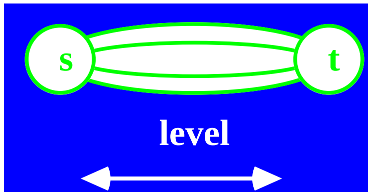
Theorem (Evan and Tarjan)

0 - 1 Flow requires min $\left(|V|^{2/3}, |E|^{1/2}\right)$ blocking steps of Dinic's Algorithm, so total time $O\left(\min\left(|V|^{2/3}, |E|^{1/2}\right)|E|\log(V)\right)$.

<u>Unit Network</u>: All capacities $\in Z$ and every vertex v other than s or t has $\begin{cases}\text{single entering edge or}\\ \text{single departing edge.}\end{cases}$

<u>Claim</u>: If Unit Network G has max flow f, then max level is $\le \left(|V| / value(f)\right) + 1$

<u>Proof</u>: G can be decomposed into $value(f)$ vertex-disjoint paths from s to t.



so value(f)• (level-1) ≤ |V|

Theorem: Dinic's Algorithm has $O\left(|V|^{1/2}\right)$ steps on unit networks.

Proof: (1) If $value(f) \le |V|^{1/2} \Rightarrow \#\, steps \le |V|^{1/2}$

(2) If $value(f) > |V|^{1/2} \Rightarrow level \le \frac{|V|}{|V|^{1/2}} + 1,$

so # steps $\le O\left(|V|^{1/2}\right)$.

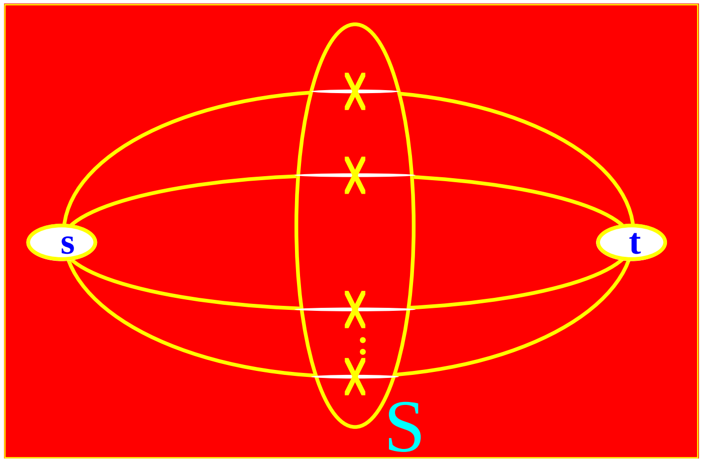
Q.E.D.

Total Time Unit Flow is $O\left(|V|^{1/2}|E|\log|E|\right)$.

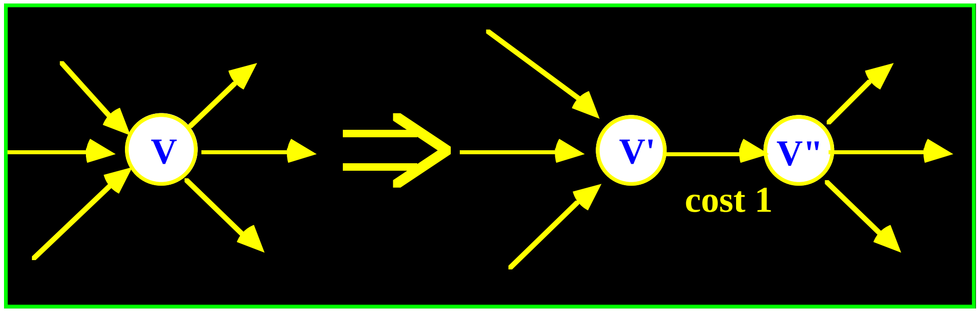
$(s-t)$ Vertex Separator $S \le V$:

if all paths from s to t contain $v \in S$.

Menger's Theroem: The size of the smallest s,t Vertex Separator S is exactly the same as the number of vertex disjoint paths from s to t.



Transform **Vertex Connectivity** to Unit **Network Flow Problem**



Total Time $O\left(|V|^{\frac{1}{2}}|E|\log(E)\right)$ to compute

s - t Vertex Connectivity $N(s,t)$ (from s to t).

$N(s,t)$ = number of disjoint paths from s to t.

$N(u,v)$ = min vertex cut size for (G,u,v)

G undirected

Vertex Connectivity: $c(G)$

$$c(G) = \begin{cases} n-1 & \text{if } G \text{ is complete graph} \\ \underset{\substack{u,v \in V \\ (u,v) \notin E}}{\underline{\text{Min}}} N(u,v) & \text{else} \end{cases}$$

Lemma

$$c(G) \le 2|E| / |V|$$

Proof

Connectivity $\le \min_{v \in V}$ **degree** v,

but $\sum_{v \in V} \text{degree}(v) = 2|E|$

Hence, $c(G) \le 2|E| / |V|$

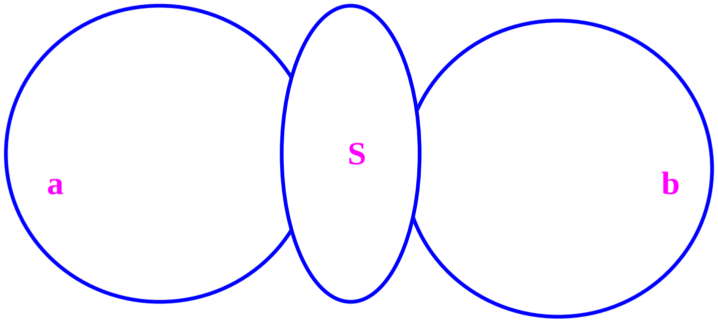
Q.E.D.

(also true for edge connectivity)

Lemma: If S is (u,v) Vertex Separator with $|S| = c(G)$, then

$$c(G) = \min_{(a,b)\notin E} N(a,b) \text{ for all } a \in V - S$$

Proof: $G - S$ has at least 2-components



Let b be any node in a component of $G-S$ which does not have a.

Thus, $N(a,b) \le |S| = c(G)$.

Q.E.D.

Idea: Choose random $a \in V$.

$$O\left(\log\left(\frac{1}{\varepsilon}\right)|V|^{3/2}|E|\log|E|\right)$$

Randomized Algorithm for Vertex Connectivity (Mehlhorn & Students)

Input: $G = (V, E)$, error bound, ε, $0 < \varepsilon < 1$

[0] $\mu \leftarrow |V| - 2$

[1] for $i = 1, 2, \ldots$ until $i \ge \log\left(\frac{1}{\varepsilon}\right) \Big/ \log\left(\frac{|V|}{\mu}\right)$

do select $a_i \in V$ at random

$\mu \leftarrow \min(\mu, \min_{b \in V} N(a_i, b))$

od

[2] output μ

Theorem: Prob$(\mu \neq c(G)) \leq \varepsilon$

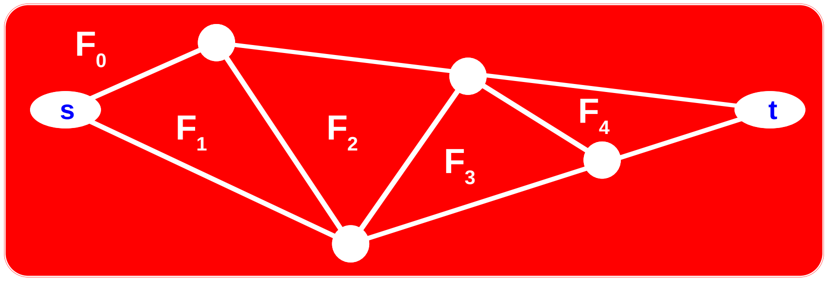
Proof: Let S be a Vertex Separator with $|S| = c(G)$. If $\mu > c(G)$, then $a_1, a_2, \ldots, a_k$ all belong to S, where

$$k \geq \frac{\log\left(1/\varepsilon\right)}{\log\left(|V|/c(G)\right)}$$

Hence, prob $(\mu > c(G)) =$ prob $(a_1, \ldots, a_k \in S)$

$$= \left(\frac{|S|}{|V|}\right)^k = \left(\frac{c(G)}{|V|}\right)^k = 2^{-\log(1/\varepsilon)} = 2^{\log \varepsilon} = \varepsilon.$$

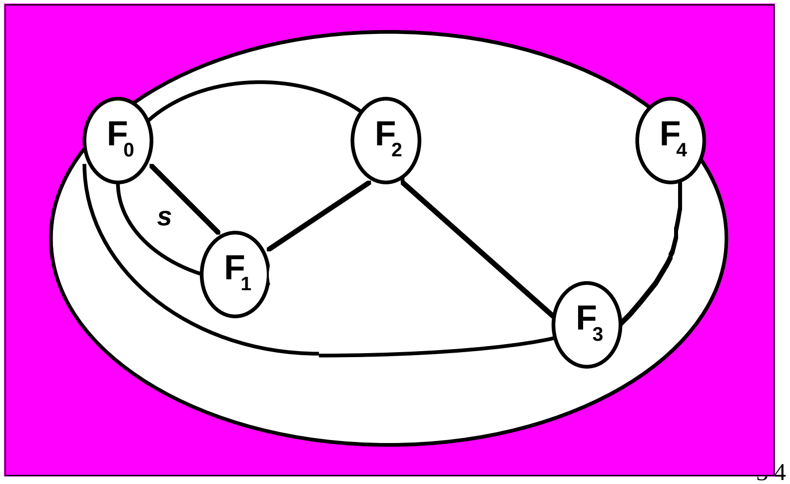
$G = (V, E)$ is a planar graph if G can be embedded on plane so no two edges cross.



Dual: $D(G) = (F, D(E))$

$F =$ faces of embedding

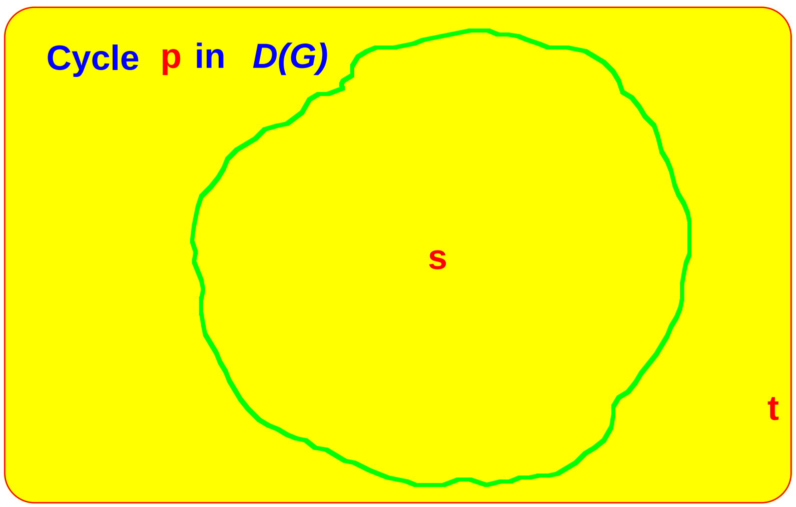
$D(E) = \{\{F_i, F_j\} | e \in E$ is between $F_i, F_j\}$



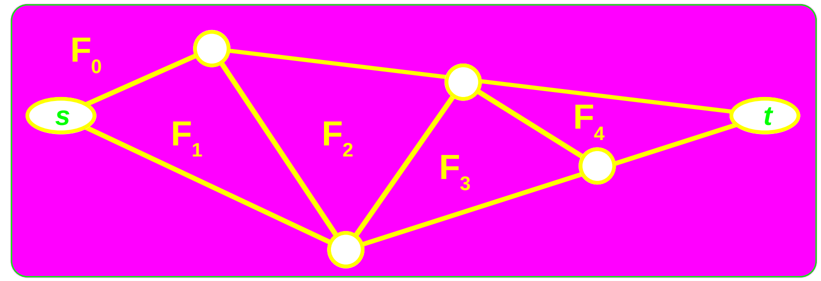
Lemma: If G is a planar embedded network, then max flow in G is same as min cost cycle in $D(G)$ separating s,t.

Proof: We assume $c(F_i, F_j) = c(e)$, if e is between F_i, F_j. Then, by min-cost cut theorem, flow value = min cut $X, \overline{X}$ between s,t

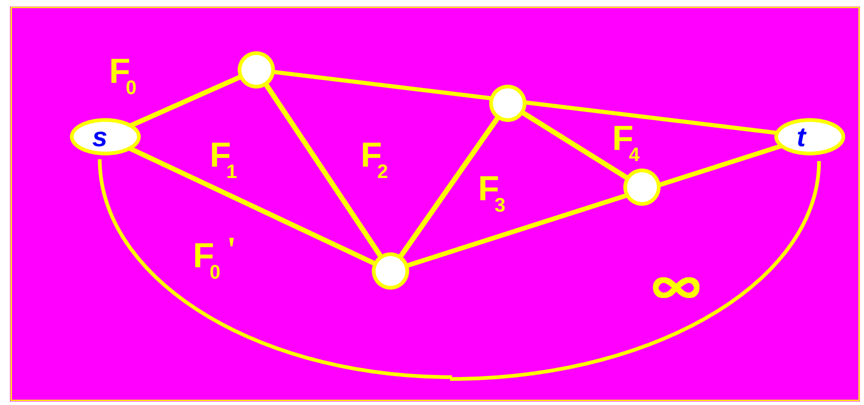
= min cost cycle in $D(G)$ separating s,t



G is outerplanar embedded if the planar embedding has face F_0 incident to all vertices.



Idea: To reduce to Min Cost Path
Add new edge (s,t) with weight ∞.



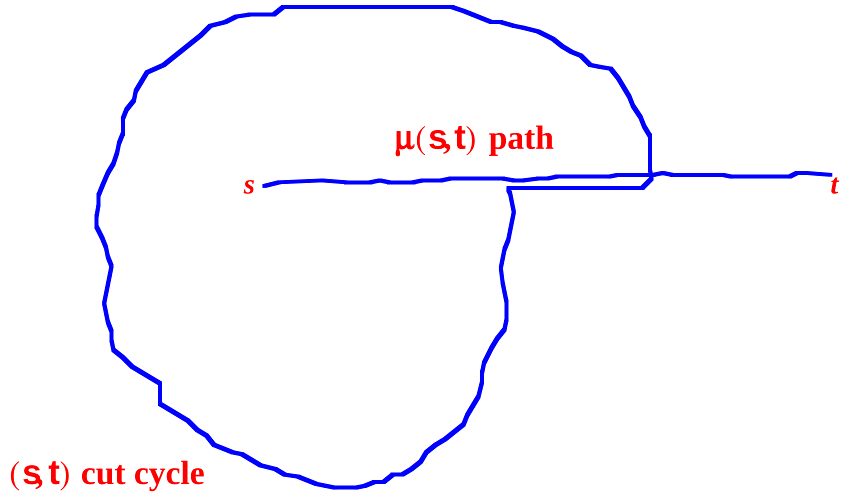
Find min cost path from F_0 to F_0©in $D(G)$

$= \text{min } s \cdot t \text{ cut in } G$

$= \text{max flow value in } G$

<u>Theorem</u>: If G is <u>outerplanar</u>, we can find max flow in $O(|V|\log|V|)$ time.

<u>Lemma</u>: [Reif] If $\mu(s,t)$ is a minimum cost path in $D(G)$ from a face bounding on s to a face bounding on t, then any min cost cycle in $D(G)$ separating s,t must contain an edge of $\mu(s,t)$.



Proof

Suppose not. Then we can shortcut any cycle of $D(G)$ separating s, t to get a shorter one, using edges of the $\mu(s,t)$ path.

Theorem: [Reif] The min cost flow in a planar graph can be computed in $O\left(|V| \log^2 |V|\right)$ time.

Proof: Idea: use $\mu(s,t)$ cut in $D(G)$ to guide a recursive divide and conquer algorithm. On each step, divide the $\mu(s,t)$ path in half and solve the problem on each half, separately, using s,t cut as separator.

Requires $O(\log|V|)$ steps

Each step $O(|V|\log|V|)$ time