Flow Network

A **flow network** is a directed graph $G = (V, E)$, and a capacity function

$$c : V \cdot V \rightarrow R^{+}$$

such that:

1. For each $(u, v) \in E$ the capacity $c(u, v) > 0$.

2. If $(u, v) \notin E$ then $c(u, v) = 0$.

3. There is a source s and a sink t.

4. Each vertex is on a directed path from s to t.

Flow

A flow in a flow network G with capacity function c is a function $f : V \cdot V \rightarrow R$ such that:

1. For all $(u, v) \in E$ (capacity constrain)

$$f(u, v) \leq c(u, v)$$

2. For all $u, v \in V$, (symmetry)

$$f(u, v) = -f(v, u).$$

3. For all $u \in V - \{s, t\}$, (flow conservation)

$$\sum_{v \in V} f(u, v) = 0.$$

The **value** of the flow is

$$|f| = \sum_{v \in V} f(s, v) = \sum_{v \in V} f(v, t).$$

Variants

Multi-source, multi-sink problem can be reduced efficiently to s single-source, single sink problem.

Multi-commodity flow requires a different approach - linear programming.

The Ford Fulkerson Method

Ford_Fulkerson_Method(G,c,s,t)

1. Initialize flow to 0;
2. While "possible"

   2.1 Improve flow (push more flow from s to t);

1. How to improve flow?
2. When to stop?

Residual Network

Given a flow f on a flow network G the **residual** capacity of an edge (u, v), $c_f(u, v)$, is the additional amount of flow that can be send from u to v before exceeding the capacity $c(u, v)$, i.e.

$$c_f(u, v) = c(u, v) - f(u, v)$$

(Note that flow is defined on pairs of vertices - not edges!)

Given a flow f on a flow network G the **residual network** G_f of G induced by f is a flow graph on G with flow capacities $c_f(u, v)$. (Not that the residual graph might have edges that are not in G!, but if (u, v) is an edge wither (u, v) or (v, u) is an edge of G.)

Improving Flow

Let f be a flow function on G.

Let f' be a flow function on the residual graph of G_f induced by f.

Defined the "flow" function $f + f'$ as

$$(f + f')(u, v) = f(u, v) + f'(u, v)$$

.

Lemma 1. *The function $f + f'$ is a flow function in G, with flow value*

$$|f + f'| = |f| + |f'|.$$

Proof.

1. For all $(u, v) \in E$,

$$\begin{array}{rcl} (f+f')(u,v) & = & f(u,v)+f'(u,v) \\ & \leq & f(u,v)+(c(u,v)-f(u,v)) \\ & = & c(u,v) \end{array}$$

2. For all $u.v \in V$,

$$\begin{array}{rcl} (f+f')(u,v) & = & f(u,v)+f'(u,v) \\ & = & -f(v,u)-f'(v,u) \\ & = & -(f+f')(v,u) \end{array}$$

3. For all $u \in V - \{s,t\}$,

$$\begin{aligned}\sum_{v\in V}(f+f')(u,v) &= \sum_{v\in V}(f(u,v)+f'(u,v)) \\ &= \sum_{v\in V} f(u,v) + \sum_{v\in V} f'(u,v)) \\ &= 0+0\end{aligned}$$

$$\begin{aligned}|f+f'| &= \sum_{v\in V}(f+f')(s,v) \\ &= \sum_{v\in V}(f(s,v)+f'(s,v)) \\ &= \sum_{v\in V} f(s,v) + \sum_{v\in V} f'(s,v)) \\ &= |f|+|f'|\end{aligned}$$

□

Given a graph $G = (V, E)$ and a flow f in G, an **augmenting path** p is a simple (directed) path connecting s to t in the residual graph G_f.

Let

$$c_f(p) = Min\{c_f(u, v) \mid (u, v) \in P\}$$

.

Lemma 2. *Let*

$$f_p(u, v) = \begin{cases} c_f(p) & (u, v) \in p \\ -c_f(p) & (v, u) \in p \\ 0 & \textit{otherwise.} \end{cases}$$

Then f_p is a flow function in G_f with value $|f_p| = c_f(p) > 0$.

Theorem 1. *If there is an augmenting path p in G_f then then there is a flow $|(f + f_p)| > |f|$ in G.*

Ford_Fulkerson_Method(G,c,s,t)

1. Initialize flow to 0;
2. While there exists an augmenting path p in G_f
   2.1 Augment the flow f with f_p;

The Max-Flow Min-Cut Theorem

An (S, T)-**cut** in a flow network $G = (V, E)$ is a partition of V to S and $T = V - S$ such that $s \in S$ and $t \in T$.

Given a flow f in G, the **net flow** across a cut (S, T) is

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v).$$

The **capacity** of the cut is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v).$$

Theorem 2. *Maximum $s - t$ flow in G equals the minimum $c(S, T)$ over all (S, T) cuts.*

Lemma 3. *Let f be a flow in G. For any (S,T) cut*

$$f(S,T) = |f|.$$

Proof.

$$\begin{aligned} f(S,T) &= f(S,V) - f(S,S) && (1)\\ &= f(S,V) && (2)\\ &= f(s,V) + f(S-s,V) && (3)\\ &= f(s,V) && (4)\\ &= |f|. && (5) \end{aligned}$$

□

Corollary 1. *The flow in G is bounded by the minimum (S,T)-cut capacity in G.*

Max-Flow Min-Cut Theorem

Theorem 3. *Let f be a flow from s to t in G. The following conditions are equivalent:*

1. f is a maximum flow.

2. The residual graph G_f allows no augmenting paths.

3. $|f| = c(S, T)$ for some (S, T)-cut.

Proof.

(1) $\Rightarrow$ (2): If G_f has an augmenting path the flow f can be improved.

(2) $\Rightarrow$ (3): We construct a an (S, T) cut as follows:

Let

$$S = \{v \in V \mid \text{there is a path from } s \text{ to } v \text{ in } G_f\}$$

and let $T = V - S$.

This partition defines an (S, T)-cut:

$s \in S$

$t \in T$ otherwise there is an augmenting path in G_f.

For each $u \in S$ and $v \in T$ we have $f(u, v) = c(u, v)$, otherwise $(u, v) \in E_f$ and $v \in S$.

$f(S, T) = c(S, T)$ and we proved that $|f| = f(S, T)$, thus

$$|f| = f(S, T) = c(S, T).$$

(3) $\Rightarrow$ (1):

We proved that for any (S, T) cut, $|f| \leq c(S, T)$, thus if $|f| = c(S, T)$ for some cut, it is a maximum flow. $\square$