

Received from Noga Alon, Feb 19, 1997

Expansion and eigenvalue gap (remark)

In [1] it is shown that the second largest eigenvalue of any d regular expander is bounded away from d . Here is a short proof of the main point in the somewhat easier fact that edge expansion implies an eigenvalue gap. This clearly supplies a certain estimate for vertex expansion as well.

Let $G = (V, E)$ be a d regular graph on n vertices denoted $\{1, 2, \dots, n\}$, let $x_1 \geq x_2 \geq \dots \geq x_n$ be nonnegative reals, and assume at least half of them are 0. Assume also that for every set U of at most half the vertices of G there are at least $c|U|$ edges between U and its complement, where c is some positive constant. Since $x_j = 0$ for all $j \geq n/2$,

$$\sum_{ij \in E} |x_i^2 - x_j^2| = \sum_{ij \in E, i < j} (x_i^2 - x_j^2) \geq \sum_{i: i < n/2} (x_i^2 - x_{i+1}^2) ci = c \sum_{i=1}^n x_i^2.$$

Therefore, by Cauchy Schwartz (twice)

$$\begin{aligned} c^2 \left(\sum_{i=1}^n x_i^2 \right)^2 &\leq \left(\sum_{ij \in E} |x_i^2 - x_j^2| \right)^2 = \left(\sum_{ij \in E} |x_i - x_j| |x_i + x_j| \right)^2 \\ &\leq \left(\sum_{ij \in E} (x_i - x_j)^2 \right) \cdot \left(\sum_{ij \in E} (x_i + x_j)^2 \right) \leq \left(\sum_{ij \in E} (x_i - x_j)^2 \right) \cdot 2d \left(\sum_{i=1}^n x_i^2 \right). \end{aligned}$$

Hence

$$\sum_{ij \in E} (x_i - x_j)^2 \geq \frac{c^2}{2d} \sum_{i=1}^n x_i^2,$$

implying (with a slight extra effort) that the second largest eigenvalue of G is at most $d - \frac{c^2}{2d}$.

References

- [1] N. Alon, Eigenvalues and expanders, *Combinatorica* 6(1986), 83-96.