7.1 Preliminaries

Recall Markov's inequality: for any nonnegative random variable X with $\mathbf{E}[X] = \mu$, we have

$$\mathbf{Pr}[X > t] \leq \frac{\mu}{t} \tag{7.1.1}$$

$$\mathbf{Pr}[X > k\mu] \leq \frac{1}{k}. \tag{7.1.2}$$

This immediately implies Chebyshev's inequality: If X is a random variable with mean μ and variance σ^2, then we have

$$\mathbf{Pr}[|X - \mu| > t] \leq \frac{\mathrm{Var}[X]}{t} \tag{7.1.3}$$

$$\mathbf{Pr}[|X - \mu| > k\sigma] \leq \frac{1}{k^2}. \tag{7.1.4}$$

7.2 Random Graphs

We will be working in the $G(n, p)$ random graph model (also known as the Erdős-Rényi model). The model defines a probability measure on graphs $G = (V, E)$ over n nodes:

$$\mathbf{Pr}[G] = \binom{n^2}{m} p^m (1-p)^{n^2-m} \tag{7.2.5}$$

(where $m = |E|$). In other words, a member of $G(n, p)$ is obtained stochastically by starting with n disconnected nodes and for each pair of nodes $x \neq y$ in G, flipping a p-biased coin, independently, to determine whether there is an edge connecting x and y. We will be looking at asymptotic properties of various families of random graphs; in this case p will be considered as a function of n.

We will use the notation $p \gg p_0$ to mean

$$\lim_{n\to\infty} \frac{p(n)}{p_0(n)} \rightarrow \infty$$

($p \ll p_0$ means $p_0 \gg p$).

It turns out that many natural properties of random graphs have a phase-transition behavior. We will focus on the property that a random graph has a K_4 subgraph:

Theorem 7.2.1 *There exists a $p_0(n)$ such that*

(i) if $p \ll p_0$ then $\mathbf{Pr}[G \in G(n,p)$ has a K_4 subgraph$] \to 0$

(ii) if $p \gg p_0$ then $\mathbf{Pr}[G \in G(n,p)$ has a K_4 subgraph$] \to 1$.

Proof: Let X be the random variable defined as the number of K_4 subgraphs in $G = (V, E)$. Then

$$X = \sum_{\substack{S \subset V \\ |S|=4}} X_S \tag{7.2.6}$$

where X_S is the indicator random variable for the event that every pair of vertices in S is connected by an edge in G. Since S has 6 pairs of vertices and the probability of any edge being present is p, we have

$$\mathbf{E}[X_S] = p^6 \tag{7.2.7}$$

$$\mathbf{E}[X] = \binom{n}{4} p^6. \tag{7.2.8}$$

Take $p_0 = n^{-2/3}$. Then we have

$$p \ll p_0 \implies \mathbf{E}[X] \to 0 \tag{7.2.9}$$

$$p \gg p_0 \implies \mathbf{E}[X] \to \infty. \tag{7.2.10}$$

The limit in (7.2.9) immediately implies (i) via Markov's inequality. However, (ii) does not immediately follow from (7.2.10), since there exist random variables whose mean grows unboundedly while the random variable goes to zero in probability. (As an example of such a random variable, consider $Y = n^2$ w.p. $\frac{1}{n}$ and $Y = 0$ w.p. $1 - \frac{1}{n}$.)

Instead, we are going to prove (ii) by bounding $\mathbf{Pr}[X = 0]$:

$$\mathbf{Pr}[X = 0] \leq \mathbf{Pr}[|X - \mu| \geq \mu] \tag{7.2.11}$$

$$\leq \frac{\mathrm{Var}[X]}{\mathbf{E}[X]^2}, \tag{7.2.12}$$

where (7.2.11) holds because the event $X = 0$ is contained in the event $|X - \mu| \geq \mu$, and the second inequality is Chebyshev's.

Let us compute

$$\mathrm{Var}[X] = \mathrm{Var}\left[\sum_S X_S\right] \tag{7.2.13}$$

$$= \sum_S \mathrm{Var}[X_S] + \sum_{S \neq T} \mathrm{Cov}(X_S, X_T). \tag{7.2.14}$$

Now $\mathrm{Var}[X_S] = \mathbf{E}[X_S] - \mathbf{E}[X_S]^2 = O(p^6)$, so $\sum \mathrm{Var}[X_S] = O(n^4 p^6)$. Recall that $\mathrm{Cov}(Y, Z) = \mathbf{E}[YZ] - \mathbf{E}[Y]\mathbf{E}[Z]$. We compute $\mathrm{Cov}(X_S, X_T)$ for three simple cases:

(a) $|S \cap T| = 0, 1$: In this case, X_S and X_T are independent, so $\mathrm{Cov}(X_S, X_T) = 0$.

(b) $|S \cap T| = 2$: In this case

$$\sum_{S \neq T} \mathrm{Cov}(X_S, X_T) = \binom{n}{6}\binom{6}{4}\binom{4}{2} p^{11} - p^{12} = O(n^6 p^{11}).$$

(c) $|S \cap T| = 3$: In this case

$$\sum_{S \neq T} \mathrm{Cov}(X_S, X_T) = O(n^5 p^9).$$

Therefore

$$\frac{\mathrm{Var}[X]}{\mathbf{E}[X]^2} = \frac{O(n^4 p^6 + n^6 p^{11} + n^5 p^9)}{O(n^4 p^6)^2} \tag{7.2.15}$$

which goes to 0 if $p \ll p_0$. This establishes (ii) and proves the theorem. ■