IQC

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Quantum Error Correction

Contents

1 Introduction

1.1 Why Error Correction?

We are not perfect (!):

- Imprecise control on the system:

$$\mathbf{U} = \mathbf{e}^{-\mathbf{i}\frac{\pi+\delta}{4}\mathbf{Z}}$$

- The system is not totally isolated

$$\mathbf{U} = \mathbf{e}^{-\mathbf{i}\epsilon\mathbf{Z}_1\mathbf{Z}_2}$$

- We are losing our qubits

$$|\boldsymbol{\Psi}\rangle \longrightarrow \text{nothing}$$

1.2 Classical Error Correction

Ingredients needed for error correction

- Error model: understanding the characteristics of errors
- Code and code words: information we want to preserve
- Conditions for error correction: how do we preserve the information

1.2.1 1 bit: classical errors

Error model (lossless)

With probability (1-p): $0 \Rightarrow 0$
With probability (1-q): $1 \Rightarrow 1$
With probability p: $0 \Rightarrow 1$
With probability q: $1 \Rightarrow 0$

Note: if we have more than one bit we have to learn about correlations between the errors

1.2.2 First example: two bit code

The model is one where we have two bits with four states (codewords):

$$\mathbf{00,\ 01,\ 10,\ 11}$$

where they get corrupted in such a way that the first qubit is not affected but the second one is:

$$\begin{aligned}
\mathbf{00} &\longrightarrow \mathbf{(1-p)00 + p01} \\
\mathbf{01} &\longrightarrow \mathbf{(1-p)01 + p00} \\
\mathbf{10} &\longrightarrow \mathbf{(1-p)10 + p11} \\
\mathbf{11} &\longrightarrow \mathbf{(1-p)11 + p10}
\end{aligned}$$

This seems to be a trivial error model, we just have to put the information in the first bit and forget about the second one. However remember this idea, as it turns out that all error correcting codes can be seen to be of that form, after we do an appropriate transformation of the codeword. In this transformed frame, the errors look trivial.

The previous example can be seen as a change of basis of the case where errors are totally correlated, i.e that the error flips both bits at the same time. In this basis the the first bit of information is the parity and the error is a parity changing operator.

$$\mathbf{00 \longrightarrow (1-p)00 + p11}$$
$$\mathbf{10 \longrightarrow (1-p)10 + p01}$$

Note that the error does preserve the parity, thus we can encode information in the logical states

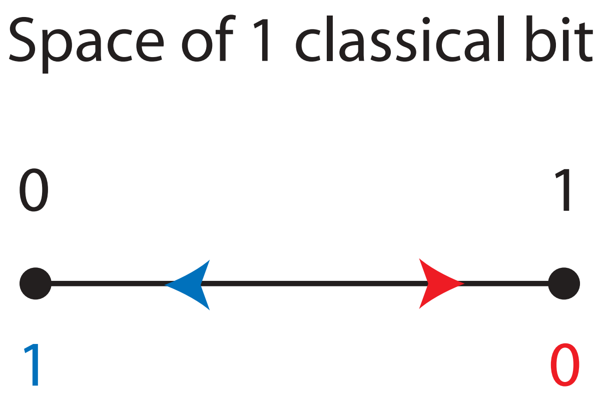
$$\mathbf{0_L = \text{Parity}\{00, 11\} \longrightarrow (1-p)00 + p11}$$
$$\mathbf{1_L = \text{Parity}\{10, 01\} \longrightarrow (1-p)10 + p01}$$

1.2.3 Independent error model

Let's assume we have a symmetric error model, independent from one bit to another

With probability (1-p): $\mathbf{0} \Rightarrow \mathbf{0}$; $\mathbf{1} \Rightarrow \mathbf{1}$

With probability p: $\mathbf{0} \Rightarrow \mathbf{1}$; $\mathbf{1} \Rightarrow \mathbf{0}$

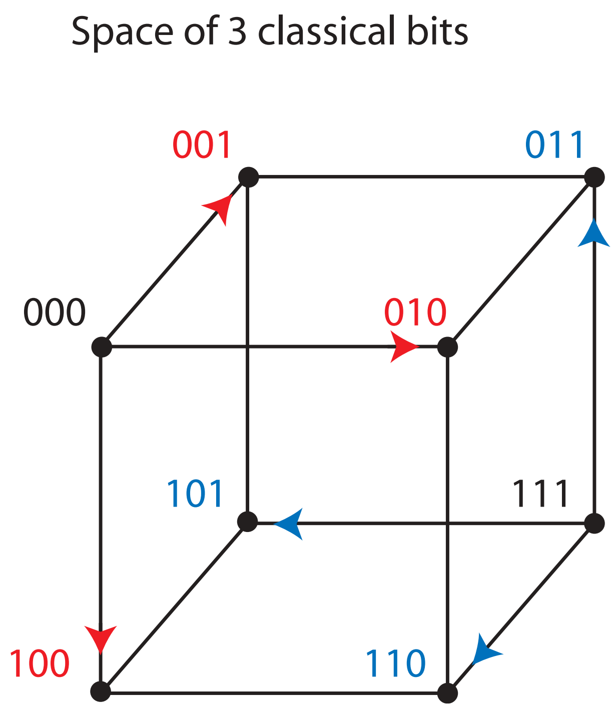


Thus if we take 3 bits and encode 0 into 000 and 1 into 111 we will have

$$000 \rightarrow \begin{cases} 000 & (1-p)^3 \\ \left.\begin{matrix} 001 \\ 010 \\ 100 \end{matrix}\right\} & p(1-p)^2 \\ \left.\begin{matrix} 011 \\ 110 \\ 101 \end{matrix}\right\} & p^2(1-p) \\ 111 & p^3 \end{cases}$$

and an analogous effect on 111.

Let's make the assumption that $p << 1$ and thus we can neglect the second order term in p. Then under the influence of the noise we have the following effect:



Note that the messages and their corresponding corrupted versions do not overlap, i.e. the 000 with corrupted version $001, 010, 100$ does not overlap with 111 or $110, 101, 011$. Thus it is possible to “undo” the effect of the noise by resetting the bits to the one obtain by taking a majority vote of the 3 bits at end. This resets $000, 001, 010, 100$ to 000 and

$111, 110, 101, 011$ to 111.

If we include the errors which occur to order p^2, we would not be able to correct them. In the next lecture we will see when it is useful to attempt the error correcting procedure.

We can identify the following elements in this error correction operations:

- the noise model
- an encoding
- an error correction operation

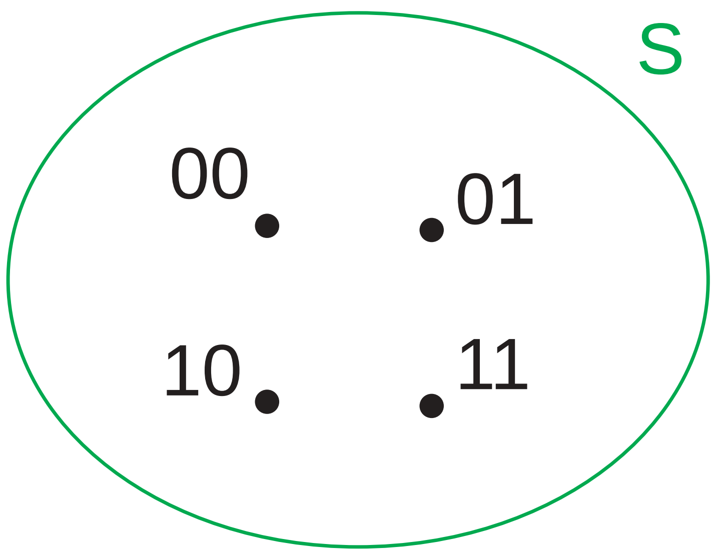
Sometimes the encoding is thought as copying the information: if this would be essential it would be impossible in the quantum world because of the no-cloning theorem.

The error correction operation could be thought as measuring the bits and taking majority, this again would not be helpful if it would be essential as it would destroy the quantum information.

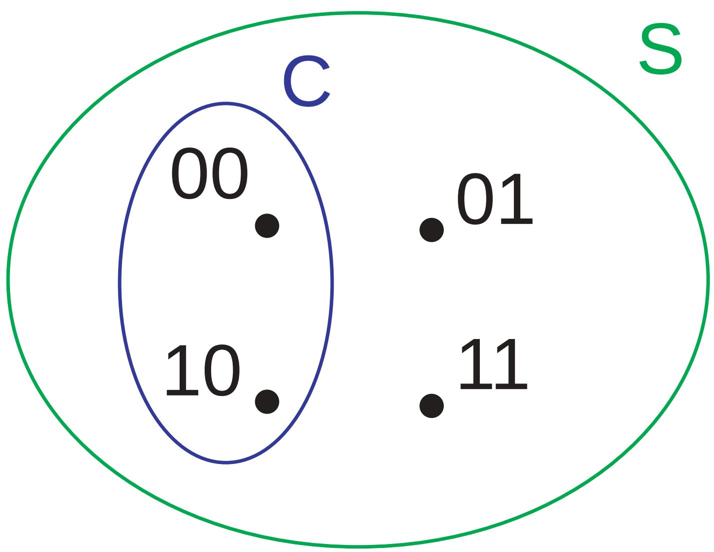
1.3 Concepts

There are a few concepts that we can now introduce:

1) The state (sometimes called word) space S, for the two bits the states are: 00, 01, 10 ,11

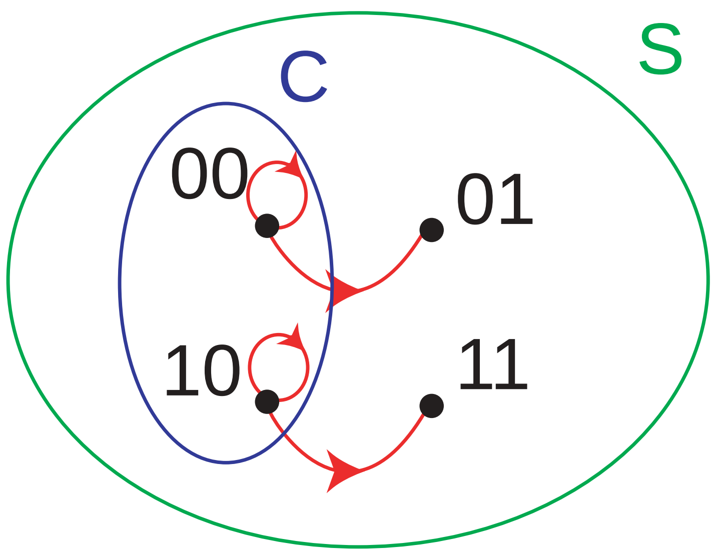


2) A code $\mathcal{C}$ is a subspace of the state (word) space.



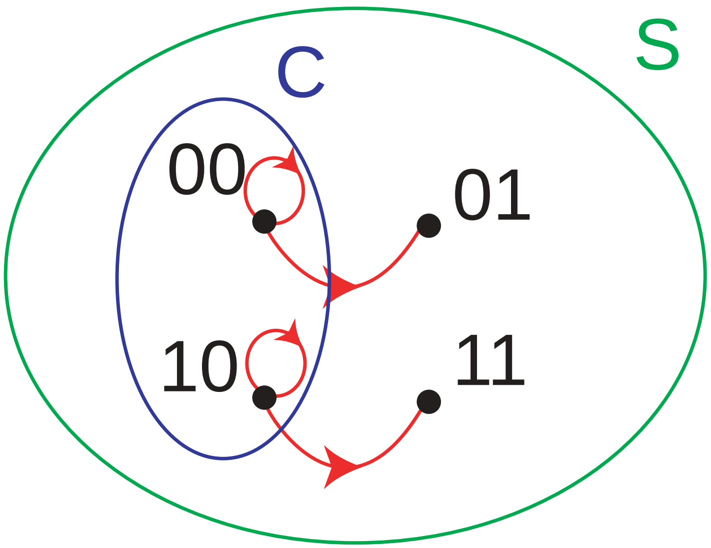
3) Error operators

We have introduced errors on the code words (we will call errors operators the map from the code words to words in the state space). We will call the trivial map of one code word to itself an “error”, for simplicity.



To detect an error we need that the effect of an error on a word give either itself or a word which does not belong to the code. ($\boldsymbol{Ex} \neq \boldsymbol{y}$, for $\boldsymbol{x} \neq \boldsymbol{y} \in \boldsymbol{\mathcal{C}}$).

For error correction we need to insure that two corrupted words are different words ($\boldsymbol{E_i x} \neq \boldsymbol{E_j y}$, for $\boldsymbol{x} \neq \boldsymbol{y} \in \boldsymbol{\mathcal{C}}$).



2 Quantum Error Correction

Contents

2.1 Error models

2.1.1 Generic 1 qubit error

A generic qubit has the state

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

but qubits might not be isolated (and now we know that there can be information hidden in quantum correlation between systems) so the most general evolution which include an environment (with state $|\epsilon\rangle$) takes the form

$$|0\rangle|\epsilon\rangle \longrightarrow |0\rangle|\epsilon_0^0\rangle + |1\rangle|\epsilon_0^1\rangle$$
$$|1\rangle|\epsilon\rangle \longrightarrow |0\rangle|\epsilon_1^0\rangle + |1\rangle|\epsilon_1^1\rangle$$

and thus

$$
\begin{aligned}
(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \rightarrow \\
& (\alpha|0\rangle + \beta|1\rangle)\frac{1}{2}(|\epsilon_0^0\rangle + |\epsilon_1^1\rangle) \quad (\Rightarrow \mathbb{1}|\Psi\rangle) \\
& +(\alpha|0\rangle - \beta|1\rangle)\frac{1}{2}(|\epsilon_0^0\rangle - |\epsilon_1^1\rangle) \quad (\Rightarrow \mathbf{Z}|\Psi\rangle) \\
& +(\alpha|1\rangle + \beta|0\rangle)\frac{1}{2}(|\epsilon_0^1\rangle + |\epsilon_1^0\rangle) \quad (\Rightarrow \mathbf{X}|\Psi\rangle) \\
& +(\alpha|1\rangle - \beta|0\rangle)\frac{1}{2}(|\epsilon_0^1\rangle - |\epsilon_1^0\rangle) \quad (\Rightarrow \mathbf{iY}|\Psi\rangle)
\end{aligned}
$$

The effect of the noise is to apply the error operators $\mathbb{1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}$ to the state $|\Psi\rangle$ depending on what the state of the environment is.

Note that these four operator form an operator basis in the acting on the 2 dimensional Hilbert space of one qubit. For n qubits we have 4^n possible operators, obtained by the tensor product of each one-qubit operator, i.e.. for two qubits we would have $\mathbb{1} \otimes \mathbb{1}, \boldsymbol{X} \otimes \mathbb{1}, \ldots, \boldsymbol{X} \otimes \boldsymbol{X}, \ldots \boldsymbol{Z} \otimes \boldsymbol{Z}$.

2.1.2 Phase shift

Let’s look at some simple examples of noise operators in physical systems such as decoherence:

$$|0\rangle|\epsilon\rangle \longrightarrow |0\rangle|\epsilon_0\rangle = |0\rangle|\epsilon\rangle$$
$$|1\rangle|\epsilon\rangle \longrightarrow |1\rangle|\epsilon_1\rangle = e^{i\theta}|1\rangle|\epsilon\rangle$$

Thus

$$(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \rightarrow (\alpha|0\rangle + e^{i\theta}\beta|1\rangle)|\epsilon\rangle$$

and which can be rewritten as

$$\begin{aligned}(\alpha|0\rangle + e^{i\theta}\beta|1\rangle)|\epsilon\rangle &= \frac{1+e^{i\theta}}{2}(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \\ &+ \frac{1-e^{i\theta}}{2}(\alpha|0\rangle - \beta|1\rangle)|\epsilon\rangle \\ &= \frac{1+e^{i\theta}}{2}\mathbb{1}(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle \\ &+ \frac{1-e^{i\theta}}{2}Z(\alpha|0\rangle + \beta|1\rangle)|\epsilon\rangle\end{aligned}$$

Here we have a certain amplitude ($\frac{1+e^{i\theta}}{2}$) of nothing happening ($\mathbb{1}$) and ($\frac{1+e^{i\theta}}{2}$) of a Z error happening.

2.1.3 Unwanted interaction with another system

Another example is a system which interacts with an environment (qubit 2) (with a coupling $U = e^{-i\theta Z_1 Z_2/2}$ previously encountered). If the second qubit starts in the state $(|0_2\rangle + |1_2\rangle)/2$, we will end up in a state

$$|0\rangle(|0_2\rangle + |1_2\rangle)/\sqrt{2} \rightarrow |0\rangle \underbrace{(e^{-i\theta/2}|0_2\rangle + e^{i\theta/2}|1_2\rangle)/\sqrt{2}}_{|\epsilon_1\rangle}$$

$$|1\rangle(|0_2\rangle + |1_2\rangle)/\sqrt{2} \rightarrow |1\rangle \underbrace{(e^{i\theta/2}|0_2\rangle + e^{-i\theta/2}|1_2\rangle)/\sqrt{2}}_{|\epsilon_2\rangle}$$

the overlap between the two states of the environment is then

$$\begin{aligned}\langle \epsilon_1 || \epsilon_2 \rangle &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta/2}\langle 0_2| + \mathrm{e}^{-\mathrm{i}\theta/2}\langle 1_2|)(\mathrm{e}^{\mathrm{i}\theta/2}|0_2\rangle + \mathrm{e}^{-\mathrm{i}\theta/2}|1_2\rangle) \\ &= \cos\theta\end{aligned}$$

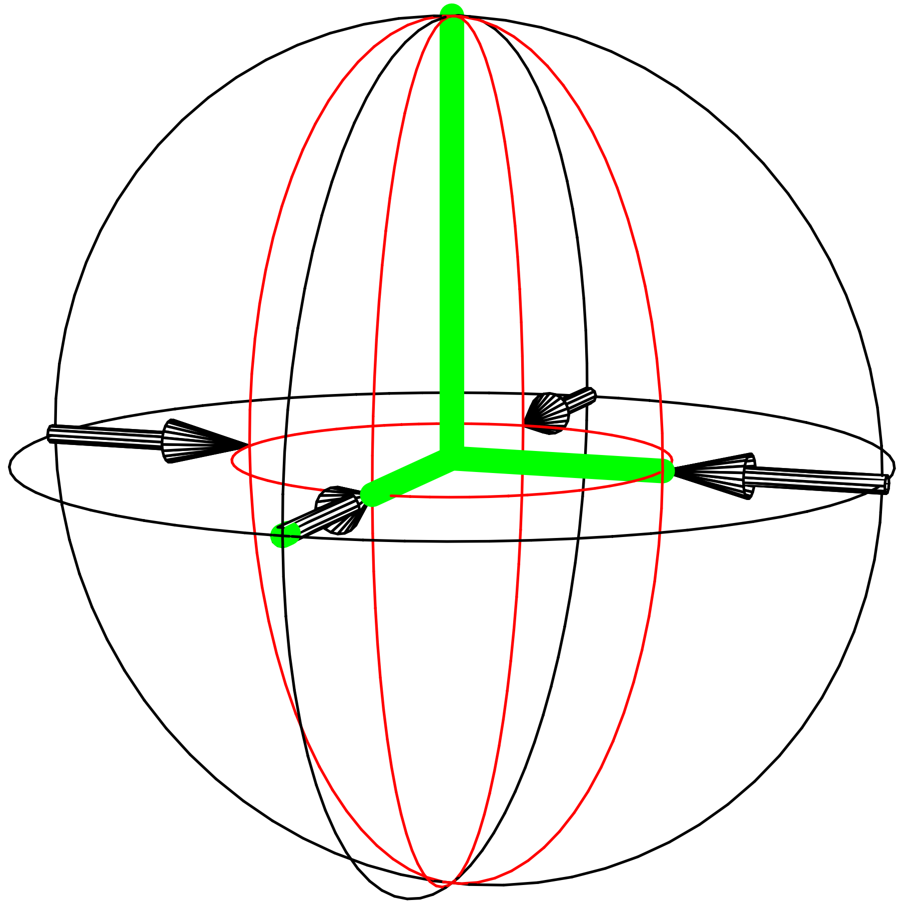
so as the interaction increases, the overlap between the environment states decrease up to $\theta = \pi/2$ when the overlap is zero.

With the state

$$|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle)$$

the density matrix for the first qubit becomes

$$\begin{aligned}\rho_1 &= \frac{1}{2}\mathrm{Tr}[\mathrm{U}|\Psi\rangle(|0_2\rangle + |1_2\rangle)(\langle 0_2| + \langle 1_2|)\langle\Psi|\mathrm{U}^\dagger] \\ &= \begin{pmatrix} \alpha\alpha^* & \alpha\beta^*\cos\theta \\ \alpha^*\beta\cos\theta & \beta\beta^* \end{pmatrix}\end{aligned}$$



The density matrix for the first qubit is:

$$
\begin{aligned}
\boldsymbol{\rho_1} &= \frac{1}{2}\mathrm{Tr}_2[\mathrm{U}|\boldsymbol{\Psi}\rangle(|\mathbf{0_2}\rangle + |\mathbf{1_2}\rangle)(\langle\mathbf{0_2}| + \langle\mathbf{1_2}|)\langle\boldsymbol{\Psi}|\mathrm{U}^\dagger] \\
&= \begin{pmatrix} \alpha\alpha^* & \alpha\beta^*\cos\theta \\ \alpha^*\beta\cos\theta & \beta\beta^* \end{pmatrix} \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta \mathrm{Z}_1/2}\begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{pmatrix}\mathrm{e}^{\mathrm{i}\theta \mathrm{Z}_1/2} \\
&\quad + \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta \mathrm{Z}_1/2}\begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{pmatrix}\mathrm{e}^{-\mathrm{i}\theta \mathrm{Z}_1/2} \\
&= \sum_i A_i \rho A_i^\dagger
\end{aligned}
$$

The A_i are called Krauss operators.