Introduction to Quantum Information Processing

Lecture 3

Michele Mosca

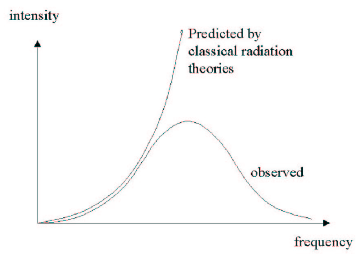
Overview

- Why quantum mechanics?
- Postulate 1: state space
- Postulate 2: unitary evolution
- Postulate 4: composite systems
- Postulate 3: measurements

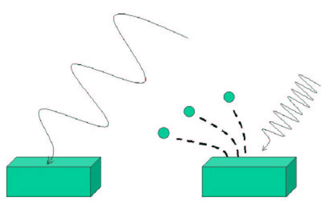
Why quantum mechanics?

- Equilibrium of radiation with the walls of a cavity
- Photoelectric effect
- Discrete spectrum of atomic radiation
- Stability of atoms

Equilibrium of radiation with the walls of a cavity?

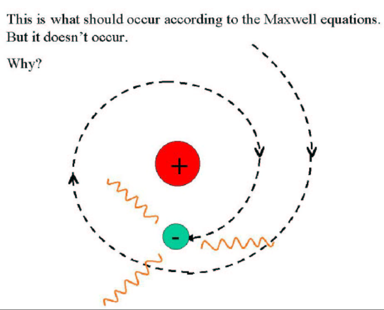


Photoelectric effect



Discrete spectrum of atomic radiation

Stability of atoms



Postulate 1: state space

Associated to any isolated physical system is a complex vector space with inner product known as the *state space* of the system. The state of the system is completely described by its *state vector*, which is a unit vector in the system's state space.

(Usually referred to as a *Hilbert space,* which is an inner product space that is complete with respect to the norm defined by the inner product. Trivial for finite dimensional complex vector spaces. We will restrict attention to finite dimensional spaces for most of this course.)

Dirac notation

For any vector $|\psi\rangle$, we let $\langle\psi|$ denote $|\psi\rangle^{\dagger}$, the complex conjugate of $|\psi\rangle$.

We denote by $\langle\varphi|\psi\rangle = \langle\varphi|\cdot|\psi\rangle$ the inner product between two vectors $|\varphi\rangle$ and $|\psi\rangle$

$\langle\psi|$ defines a linear function that maps

$|\varphi\rangle \rightarrow \langle\psi|\varphi\rangle$ (I.e. $\langle\psi|(|\varphi\rangle) = \langle\psi|\varphi\rangle$ … it maps any state $|\varphi\rangle$ to the coefficient of its $|\psi\rangle$ component)

Postulate 2: evolution

The evolution of a closed quantum system is described by a unitary transformation.
That is, the state $|\psi(t_1)\rangle$ of the system at time t_1 is related to the state $|\psi(t_2)\rangle$ at time t_2 by a unitary operator U that only depends on t_1 and t_2.

$$|\psi(t_2)\rangle = U(t_1, t_2)|\psi(t_1)\rangle$$

(if we want a linear evolution that preserves the norm, then we must have unitary evolution)

More Dirac notation

$|\psi\rangle\langle\psi|$ defines a linear operator that maps

$$|\psi\rangle\langle\psi||\varphi\rangle \rightarrow |\psi\rangle\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle$$

(I.e. projects a state to its $|\psi\rangle$ component)

(Aside: this projection operator also corresponds to the "density matrix" for $|\psi\rangle$)

More generally, we can also have operators like $|\theta\rangle\langle\psi|$

$$|\theta\rangle\langle\psi||\varphi\rangle \rightarrow |\theta\rangle\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle|\theta\rangle$$

More Dirac notation

For example, the one qubit NOT gate corresponds to the operator $|0\rangle\langle 1| + |1\rangle\langle 0|$

e.g. $(|0\rangle\langle 1| + |1\rangle\langle 0|)(|0\rangle)$

$$= |0\rangle\langle 1||0\rangle + |1\rangle\langle 0||0\rangle$$

$$= |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle$$

$$= 0|0\rangle + 1|1\rangle$$

$$= |1\rangle$$

The NOT gate is a 1-qubit unitary operation.

Special unitaries: Pauli Matrices

The NOT operation, is often called the X or σ_X operation.

$$X = \sigma_X = NOT = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Z = \sigma_Z = signflip = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Y = \sigma_Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Special unitaries: Pauli Matrices

- An operator H is Hermitian if $H = H^\dagger$ († means conjugate and transpose). The Pauli matrices are Hermitian operators.
- An operator U is unitary if $U^{-1} = U^\dagger$. Operators of the form

$$U = \mathrm{e}^{-iH}$$

are unitary. E.g., using the Pauli matrices

$$U = \mathrm{e}^{-i\theta\vec{n}\cdot\vec{\sigma}} = \mathrm{e}^{-i\theta(n_x X + n_y Y + n_z Z)} = \cos\theta \mathbb{1} - i\sin\theta\, \vec{n}\cdot\vec{\sigma}$$

for $\vec{n}$ being a unit vector.

What is e^{iHt} ??

It helps to start with the spectral decomposition theorem.

Spectral decomposition

- Definition: an operator (or matrix) M is "normal" if $MM^{\dagger}=M^{\dagger}M$
- E.g. Unitary matrices U satisfy $UU^{\dagger}=U^{\dagger}U=I$
- E.g. Density matrices (since they satisfy $\rho=\rho^{\dagger}$; i.e. "Hermitian") are also normal

Spectral decomposition

- Theorem: For any normal matrix M, there is a unitary matrix P so that

$M=P\Lambda P^{\dagger}$ where Λ is a diagonal matrix.

- The diagonal entries of Λ are the eigenvalues. The columns of P encode the eigenvectors.

e.g. NOT gate

$X|0\rangle=|1\rangle \qquad X|1\rangle=|0\rangle \qquad X=|0\rangle\langle 1|+|1\rangle\langle 0|$

$$[X]_{\{|0\rangle,|1\rangle\}}=\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{bmatrix}$$

$|+\rangle=\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle=\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle$

$X|+\rangle=|+\rangle \qquad X|-\rangle=-|-\rangle \qquad X=|+\rangle\langle +|-|-\rangle\langle -|$

$$[X]_{\{|+\rangle,|-\rangle\}}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

Spectral decomposition

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & \cdots & |\psi_n\rangle \end{bmatrix}$$

Spectral decomposition

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Spectral decomposition

$$P^\dagger = \begin{bmatrix} a_{11}^* & a_{21}^* & \cdots & a_{n1}^* \\ a_{12}^* & a_{22}^* & \cdots & a_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}^* & a_{2n}^* & \cdots & a_{nn}^* \end{bmatrix} = \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_n| \end{bmatrix}$$

Spectral decomposition

$$P\Lambda P^\dagger$$

$$= \left[|\psi_1\rangle \quad |\psi_2\rangle \quad \cdots \quad |\psi_n\rangle \right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_n| \end{bmatrix}$$

$$= \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$\left(\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \sum_i \lambda_i \begin{bmatrix} \ddots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \end{bmatrix} \leftarrow i^{th} \text{ row} \atop \qquad\qquad\qquad \uparrow \; i^{th} \text{ column} \right)$$

Verifying eigenvectors and eigenvalues

$$P\Lambda P^\dagger |\psi_2\rangle$$

$$= \left[|\psi_1\rangle \quad |\psi_2\rangle \quad \cdots \quad |\psi_n\rangle \right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_n| \end{bmatrix} |\psi_2\rangle$$

$$= \left[|\psi_1\rangle \quad |\psi_2\rangle \quad \cdots \quad |\psi_n\rangle \right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \langle\psi_1|\psi_2\rangle \\ \langle\psi_2|\psi_2\rangle \\ \vdots \\ \langle\psi_n|\psi_2\rangle \end{bmatrix}$$

Verifying eigenvectors and eigenvalues

$$= \left[|\psi_1\rangle \quad |\psi_2\rangle \quad \cdots \quad |\psi_n\rangle \right] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \left[|\psi_1\rangle \quad |\psi_2\rangle \quad \cdots \quad |\psi_n\rangle \right] \begin{bmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{bmatrix} = \lambda_2 |\psi_2\rangle$$

Why is spectral decomposition useful?

Note that $\left(|\psi_i\rangle\langle\psi_i|\right)^m = |\psi_i\rangle\langle\psi_i|$ recall $\langle\psi_i|\psi_j\rangle = \delta_{ij}$

So $\left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|\right)^m = \sum_i \lambda_i^m |\psi_i\rangle\langle\psi_i|$

Consider $f(x) = \sum_m a_m x^m$ e.g. $e^x = \sum_m \frac{1}{m!} x^m$

Why is spectral decomposition useful?

$$f(M) = \sum_m a_m M^m = \sum_m a_m \left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|\right)^m$$

$$= \sum_m a_m \sum_i \lambda_i^m |\psi_i\rangle\langle\psi_i| = \sum_i \left(\sum_m a_m \lambda_i^m\right) |\psi_i\rangle\langle\psi_i|$$

$$= \sum_i f(\lambda_i) |\psi_i\rangle\langle\psi_i|$$

Same thing in matrix notation

$$f(M) = \sum_m a_m M^m$$

$$f(P\Lambda P^\dagger) = \sum_m a_m (P\Lambda P^\dagger)^m = \sum_m a_m P\Lambda^m P^\dagger = P\left(\sum_m a_m \Lambda^m\right)P^\dagger$$

$$= P\left(\sum_m a_m \begin{bmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{bmatrix}\right)P^\dagger = P\begin{bmatrix} \sum_m a_m \lambda_1^m & & \\ & \ddots & \\ & & \sum_m a_m \lambda_n^m \end{bmatrix}P^\dagger$$

Same thing in matrix notation

$$f(P\Lambda P^\dagger) = P\begin{bmatrix} \sum_m a_m \lambda_1^m & & \\ & \ddots & \\ & & \sum_m a_m \lambda_n^m \end{bmatrix} P^\dagger$$

$$= P\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} P^\dagger$$

$$= \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & \cdots & |\psi_n\rangle \end{bmatrix} \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_n| \end{bmatrix}$$

Same thing in matrix notation

$$f(P\Lambda P^\dagger) = P\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} P^\dagger$$

$$= \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & \cdots & |\psi_n\rangle \end{bmatrix} \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_n| \end{bmatrix}$$

$$= \sum_i f(\lambda_i)|\psi_i\rangle\langle\psi_i|$$

Schroedinger equation

Postulate 2: The evolution of an isolated quantum system is given by the Schrödinger equation

$$-i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \quad (12)$$

where H is an operator called the Hamiltonian which defines the theory that we are working with (electromagnetism, QCD, gravity, string theory . . .).

There is a formal solution for this equation

$$|\Psi(t)\rangle = e^{-i\int dtH}|\Psi(0)\rangle \quad (13)$$

If H is hermitian, $e^{-i\int dtH}$ is a unitary operator that we will call U. In quantum computation, U is a representation of the algorithm.

Postulate 4: composite systems

A system S that is composed entirely of two subsystems, A and B, with state spaces H_A and H_B respectively, has state space $H_A \otimes H_B$

A system S that is composed entire of several subsystems, A_1,A_2, …, A_n with state spaces H_{A1}, H_{A2}, … respectively, has state space $H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n}$

Postulate 3: measurements

"Von Neumann measurement in the computational basis"

- Suppose we have a universal set of quantum gates, and the ability to measure each qubit in the basis $\{|0\rangle, |1\rangle\}$
- If we measure $|\Phi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle)$ we get $|b\rangle$ with probability $|\alpha_b|^2$

In section 2.2.5, this is described as follows

- We have the projection operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ satisfying $P_0 + P_1 = I$
- We consider the projection operator or "observable" $M = 0P_0 + 1P_1 = P_1$
- Note that 0 and 1 are the eigenvalues
- When we measure this observable M, the probability of getting the eigenvalue b is $\Pr(b) = \langle \Phi | P_b | \Phi \rangle = |\alpha_b|^2$ and we are in that case left with the state $\frac{P_b|\Phi\rangle}{\sqrt{p(b)}} = \frac{\alpha_b}{|\alpha_b|}|b\rangle \approx |b\rangle$