A LECTURE ON GROVER'S QUANTUM SEARCH ALGORITHM VERSION 1.1

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Abstract. This paper ia a written version of a one hour lecture given on Lov Grover's quantum database search algorithm. It is based on [4], [5], and [9].

Contents

1. Problem definition

We consider the problem of searching an unstructured database of $N = 2^n$ records for exactly one record which has been specifically marked. This can be rephrased in mathematical terms as an oracle problem as follows:

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Label the records of the database with the integers

$$0, 1, 2, \ \ldots \ , N-1 \ ,$$

and denote the label of the unknown marked record by x_0. We are given an oracle which computes the n bit binary function

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

defined by

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ \\ 0 & \text{otherwise} \end{cases}$$

We remind the readers that, as a standard oracle idealization, we have no access to the internal workings of the function f. It operates simply as a blackbox function, which we can query as many times as we like. But with each such a query comes an associated computational cost.

Search Problem for an Unstructured Database. *Find the record labeled as x_0 with the minimum amount of computational work, i.e., with the minimum number of queries of the oracle f.*

From probability theory, we know that if we examine k records, i.e., if we compute the oracle f for k randomly chosen records, then the probability of finding the record labeled as x_0 is k/N. Hence, on a classical computer it takes $O(N) = O(2^n)$ queries to find the record labeled x_0.

2. The quantum mechanical perspective

However, as Lov Grover so astutely observed, on a quantum computer the search of an unstructured database can be accomplished in $O(\sqrt{N})$ steps, or more precisely, with the application of $O(\sqrt{N} \lg N)$ sufficiently local unitary transformations. Although this is not exponentially faster, it is a significant speedup.

Let $\mathcal{H}_2$ be a 2 dimensional Hilbert space with orthonormal basis

$$\{|0\rangle, |1\rangle\} \ ;$$

and let

$$\{|0\rangle, |1\rangle, \ \ldots \ , |N-1\rangle\}$$

denote the induced orthonormal basis of the Hilbert space

$$\mathcal{H} = \bigotimes_{0}^{N-1} \mathcal{H}_2 \ .$$

From the quantum mechanical perspective, the oracle function f is given as a blackbox unitary transformation U_f, i.e., by

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{H}_2 & \stackrel{U_f}{\longrightarrow} & \mathcal{H} \otimes \mathcal{H}_2 \\ |x\rangle \otimes |y\rangle & \longmapsto & |x\rangle \otimes |f(x) \oplus y\rangle \end{array}$$

where '$\oplus$' denotes exclusive 'OR', i.e., addition modulo 2.[1]

Instead of U_f, we will use the computationally equivalent unitary transformation

$$I_{|x_0\rangle}\left(|x\rangle\right) = (-1)^{f(x)}\,|x\rangle = \begin{cases} -\,|x_0\rangle & \text{if } \ x = x_0 \\ \\ \ \ |x\rangle & \text{otherwise} \end{cases}$$

That $I_{|x_0\rangle}$ is computationally equivalent to U_f follows from the easily verifiable fact that

$$U_f\left(|x\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \left(I_{|x_0\rangle}\left(|x\rangle\right)\right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \ ,$$

and also from the fact that U_f can be constructed from a controlled-$I_{|x_0\rangle}$ and two one qubit Hadamard transforms. (For details, please refer to [10], [11].)

The unitary transformation $I_{|x_0\rangle}$ is actually an **inversion** [1] in $\mathcal{H}$ about the hyperplane perpendicular to $|x_0\rangle$. This becomes evident when $I_{|x_0\rangle}$ is rewritten in the form

$$I_{|x_0\rangle} = I - 2\,|x_0\rangle\,\langle x_0| \ \ ,$$

where 'I' denotes the identity transformation. More generally, for any unit length ket $|\psi\rangle$, the unitary transformation

$$I_{|\psi\rangle} = I - 2\,|\psi\rangle\,\langle\psi|$$

is an inversion in $\mathcal{H}$ about the hyperplane orthogonal to $|\psi\rangle$.

[1]Please note that $U_f = (\nu \circ \iota)\,(f)$, as defined in sections 10.3 and 10.4 of [12].

3. Properties of the inversion $I_{|\psi\rangle}$

We digress for a moment to discuss the properties of the unitary transformation $I_{|\psi\rangle}$. To do so, we need the following definition.

Definition 1. *Let $|\psi\rangle$ and $|\chi\rangle$ be two kets in $\mathcal{H}$ for which the bracket product $\langle\psi \mid \chi\rangle$ is a real number. We define*

$$\mathcal{S}_{\mathbb{C}} = Span_{\mathbb{C}}\left(|\psi\rangle, |\chi\rangle\right) = \{\alpha|\psi\rangle + \beta|\chi\rangle \in \mathcal{H} \mid \alpha, \beta \in \mathbb{C}\}$$

as the sub-Hilbert space of $\mathcal{H}$ spanned by $|\psi\rangle$ and $|\chi\rangle$. We associate with the Hilbert space $\mathcal{S}_{\mathbb{C}}$ a real inner product space lying in $\mathcal{S}_{\mathbb{C}}$ defined by

$$\mathcal{S}_{\mathbb{R}} = Span_{\mathbb{R}}\left(|\psi\rangle, |\chi\rangle\right) = \{a|\psi\rangle + b|\chi\rangle \in \mathcal{H} \mid a, b \in \mathbb{R}\} \ ,$$

where the inner product on $\mathcal{S}_{\mathbb{R}}$ is that induced by the bracket product on $\mathcal{H}$. If $|\psi\rangle$ and $|\chi\rangle$ are also linearly independent, then $\mathcal{S}_{\mathbb{R}}$ is a 2 dimensional real inner product space (i.e., the 2 dimensional Euclidean plane) lying inside of the complex 2 dimensional space $\mathcal{S}_{\mathbb{C}}$.

Proposition 1. *Let $|\psi\rangle$ and $|\chi\rangle$ be two linearly independent unit length kets in $\mathcal{H}$ with real bracket product; and let $\mathcal{S}_{\mathbb{C}} = Span_{\mathbb{C}}\left(|\psi\rangle, |\chi\rangle\right)$ and $\mathcal{S}_{\mathbb{R}} = Span_{\mathbb{R}}\left(|\psi\rangle, |\chi\rangle\right)$. Then*

1) *Both $\mathcal{S}_{\mathbb{C}}$ and $\mathcal{S}_{\mathbb{R}}$ are invariant under the transformations $I_{|\psi\rangle}$, $I_{|\chi\rangle}$, and hence $I_{|\psi\rangle} \circ I_{|\chi\rangle}$, i.e.,*

$I_{\|\psi\rangle}(\mathcal{S}_{\mathbb{C}}) = \mathcal{S}_{\mathbb{C}}$	*and*	$I_{\|\psi\rangle}(\mathcal{S}_{\mathbb{R}}) = \mathcal{S}_{\mathbb{R}}$
$I_{\|\chi\rangle}(\mathcal{S}_{\mathbb{C}}) = \mathcal{S}_{\mathbb{C}}$	*and*	$I_{\|\chi\rangle}(\mathcal{S}_{\mathbb{R}}) = \mathcal{S}_{\mathbb{R}}$
$I_{\|\psi\rangle} I_{\|\chi\rangle}(\mathcal{S}_{\mathbb{C}}) = \mathcal{S}_{\mathbb{C}}$	*and*	$I_{\|\psi\rangle} I_{\|\chi\rangle}(\mathcal{S}_{\mathbb{R}}) = \mathcal{S}_{\mathbb{R}}$

2) *If $L_{\left|\psi^{\perp}\right\rangle}$ is the line in the plane $\mathcal{S}_{\mathbb{R}}$ which passes through the origin and which is perpendicular to $|\psi\rangle$, then $I_{|\psi\rangle}$ restricted to $\mathcal{S}_{\mathbb{R}}$ is a reflection in (i.e., a Möbius inversion [1] about) the line $L_{\left|\psi^{\perp}\right\rangle}$. A similar statement can be made in regard to $|\chi\rangle$.*
3) *If $\left|\psi^{\perp}\right\rangle$ is a unit length vector in $\mathcal{S}_{\mathbb{R}}$ perpendicular to $|\psi\rangle$, then*

$$-I_{|\psi\rangle} = I_{\left|\psi^{\perp}\right\rangle} \ .$$

(Hence, $\left\langle\psi^{\perp} \mid \chi\right\rangle$ is real.)

Finally we note that, since $I_{|\psi\rangle} = I - 2\,|\psi\rangle\,\langle\psi|$, it follows that

Proposition 2. *If $|\psi\rangle$ is a unit length ket in $\mathcal{H}$, and if U is a unitary transformation on $\mathcal{H}$, then*

$$UI_{|\psi\rangle}U^{-1} = I_{U|\psi\rangle} \ .$$

4. The method in Lov's "madness"

Let $H : \mathcal{H} \longrightarrow \mathcal{H}$ be the Hadamard transform, i.e.,

$$H = \bigotimes_{0}^{n-1} H^{(2)} \ ,$$

where

$$H^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with respect to the basis $|0\rangle$, $|1\rangle$.

We begin by using the Hadamard transform H to construct a state $|\psi_0\rangle$ which is an equal superposition of all the standard basis states $|0\rangle$, $|1\rangle$,...,$|N-1\rangle$ (including the unknown state $|x_0\rangle$), i.e.,

$$|\psi_0\rangle = H\,|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle \ .$$

Both $|\psi_0\rangle$ and the unknown state $|x_0\rangle$ lie in the Euclidean plane $\mathcal{S}_{\mathbb{R}} = Span_{\mathbb{R}}\left(|\psi_0\rangle, |x_0\rangle\right)$. Our strategy is to rotate within the plane $\mathcal{S}_{\mathbb{R}}$ the state $|\psi_0\rangle$ about the origin until it is as close as possible to $|x_0\rangle$. Then a measurement with respect to the standard basis of the state resulting from rotating $|\psi_0\rangle$, will produce $|x_0\rangle$ with high probability.

To achieve this objective, we use the oracle $I_{|x_0\rangle}$ to construct the unitary transformation

$$Q = -HI_{|0\rangle}H^{-1}I_{|x_0\rangle} \ ,$$

which by proposition 2 above, can be reexpressed as

$$Q = -I_{|\psi_0\rangle}I_{|x_0\rangle} \ .$$

Let $\left|x_0^{\perp}\right\rangle$ and $\left|\psi_0^{\perp}\right\rangle$ denote unit length vectors in $\mathcal{S}_{\mathbb{R}}$ perpendicular to $|x_0\rangle$ and $|\psi_0\rangle$, respectively. There are two possible choices for each of $\left|x_0^{\perp}\right\rangle$

and $\left|\psi_0^{\perp}\right\rangle$ respectively. To remove this minor, but nonetheless annoying, ambiguity, we select $\left|x_0^{\perp}\right\rangle$ and $\left|\psi_0^{\perp}\right\rangle$ so that the orientation of the plane $\mathcal{S}_{\mathbb{R}}$ induced by the ordered spanning vectors $|\psi_0\rangle$, $|x_0\rangle$ is the same orientation as that induced by each of the ordered bases $\left|x_0^{\perp}\right\rangle$, $|x_0\rangle$ and $|\psi_0\rangle$, $\left|\psi_0^{\perp}\right\rangle$. (Please refer to Figure 2.)

Remark 1. *The removal of the above ambiguities is really not essential. However, it does simplify the exposition given below.*

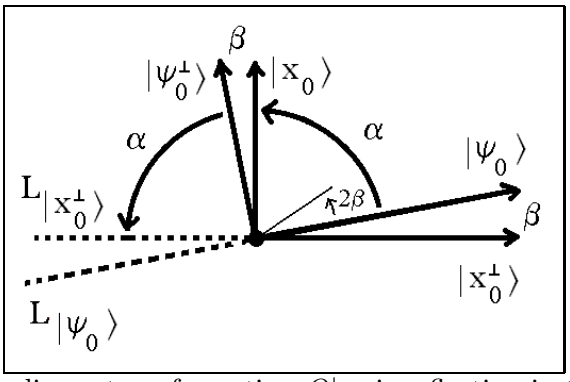


Figure 2. The linear transformation $Q|_{\mathcal{S}_{\mathbb{R}}}$ is reflection in the line $L_{|x_0^{\perp}\rangle}$ followed by reflection in the line $L_{|\psi_0\rangle}$ which is the same as rotation by the angle 2β. Thus, $Q|_{\mathcal{S}_{\mathbb{R}}}$ rotates $|\psi_0\rangle$ by the angle 2β toward $|x_0\rangle$.

We proceed by noting that, by the above proposition 1, the plane $\mathcal{S}_{\mathbb{R}}$ lying in $\mathcal{H}$ is invariant under the linear transformation Q, and that, when Q is restricted to the plane $\mathcal{S}_{\mathbb{R}}$, it can be written as the composition of two inversions, i.e.,

$$Q|_{\mathcal{S}_{\mathbb{R}}} = I_{\left|\psi_0^{\perp}\right\rangle} I_{|x_0\rangle} \ .$$

In particular, $Q|_{\mathcal{S}_{\mathbb{R}}}$ is the composition of two inversions in $\mathcal{S}_{\mathbb{R}}$, the first in the line $L_{|x_0^{\perp}\rangle}$ in $\mathcal{S}_{\mathbb{R}}$ passing through the origin having $|x_0\rangle$ as normal, the second in the line $L_{|\psi_0\rangle}$ through the origin having $\left|\psi_0^{\perp}\right\rangle$ as normal.[2]

We can now apply the following theorem from plane geometry:

[2]The line $L_{|x_0^{\perp}\rangle}$ is the intersection of the plane $\mathcal{S}_{\mathbb{R}}$ with the hyperplane in $\mathcal{H}$ orthogonal to $|x_0\rangle$. A similar statement can be made in regard to $L_{|\psi_0\rangle}$.

Theorem 1. *If L_1 and L_2 are lines in the Euclidean plane $\mathbb{R}^2$ intersecting at a point O; and if β is the angle in the plane from L_1 to L_2, then the operation of reflection in L_1 followed by reflection in L_2 is just rotation by angle 2β about the point O.*

Let β denote the angle in $S_{\mathbb{R}}$ from $L_{\left|x_0^{\perp}\right\rangle}$ to $L_{|\psi_0\rangle}$, which by plane geometry is the same as the angle from $\left|x_0^{\perp}\right\rangle$ to $|\psi_0\rangle$, which in turn is the same as the angle from $|x_0\rangle$ to $\left|\psi_0^{\perp}\right\rangle$. Then by the above theorem $Q|_{\mathcal{S}_{\mathbb{R}}} = I_{\left|\psi_0^{\perp}\right\rangle} I_{|x_0\rangle}$ is a rotation about the origin by the angle 2β.

The key idea in Grover's algorithm is to move $|\psi_0\rangle$ toward the unknown state $|x_0\rangle$ by successively applying the rotation Q to $|\psi_0\rangle$ to rotate it around to $|x_0\rangle$. This process is called **amplitude amplification**. Once this process is completed, the measurement of the resulting state (with respect to the standard basis) will, with high probability, yield the unknown state $|x_0\rangle$. This is the essence of Grover's algorithm.

But how many times K should we apply the rotation Q to $|\psi_0\rangle$? If we applied Q too many or too few times, we would over- or undershoot our target state $|x_0\rangle$.

We determine the integer K as follows:

Since

$$|\psi_0\rangle = \sin\beta\,|x_0\rangle + \cos\beta\left|x_0^{\perp}\right\rangle \ ,$$

the state resulting after k applications of Q is

$$|\psi_k\rangle = Q^k\,|\psi_0\rangle = \sin\left[(2k+1)\,\beta\right]|x_0\rangle + \cos\left[(2k+1)\,\beta\right]\left|x_0^{\perp}\right\rangle \ .$$

Thus, we seek to find the smallest positive integer $K = k$ such that

$$\sin\left[(2k+1)\,\beta\right]$$

is as close as possible to 1. In other words, we seek to find the smallest positive integer $K = k$ such that

$$(2k+1)\,\beta$$

is as close as possible to $\pi/2$. It follows that[3]

$$K = k = round\left(\frac{\pi}{4\beta} - \frac{1}{2}\right) \ ,$$

where "*round*" is the function that rounds to the nearest integer.

We can determine the angle β by noting that the angle α from $|\psi_0\rangle$ and $|x_0\rangle$ is complementary to β, i.e.,

$$\alpha + \beta = \pi/2 \ ,$$

and hence,

$$\frac{1}{\sqrt{N}} = \langle x_0 \mid \psi_0 \rangle = \cos\alpha = \cos(\frac{\pi}{2} - \beta) = \sin\beta \ .$$

Thus, the angle β is given by

$$\beta = \sin^{-1}\left(\frac{1}{\sqrt{N}}\right) \approx \frac{1}{\sqrt{N}} \ \text{ (for large } N) \ ,$$

and hence,

$$K = k = round\left(\frac{\pi}{4\sin^{-1}\left(\frac{1}{\sqrt{N}}\right)} - \frac{1}{2}\right) \approx round\left(\frac{\pi}{4}\sqrt{N} - \frac{1}{2}\right) \ \text{(for large } N).$$

5. Summary of Grover's algorithm

In summary, we provide the following outline of Grover's algorithm:

[3]The reader may prefer to use the *floor* function instead of the *round* function.

Grover's Algorithm

$\boxed{\mathbb{STEP}\ 0.}$ (Initialization)

$$|\psi\rangle \longleftarrow H|0\rangle = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}|j\rangle$$
$$k \longleftarrow 0$$

$\boxed{\mathbb{STEP}\ 1.}$ Loop until $k = round\left(\frac{\pi}{4\sin^{-1}\left(1/\sqrt{N}\right)} - \frac{1}{2}\right) \approx round\left(\frac{\pi}{4}\sqrt{N} - \frac{1}{2}\right)$

$$|\psi\rangle \longleftarrow Q|\psi\rangle = -HI_{|0\rangle}HI_{|x_0\rangle}|\psi\rangle$$
$$k \longleftarrow k+1$$

$\boxed{\mathbb{STEP}\ 2.}$ Measure $|\psi\rangle$ with respect to the standard basis $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ to obtain the marked unknown state $|x_0\rangle$ with probability $\geq 1 - \frac{1}{N}$.

We complete our summary with the following theorem:

Theorem 2. *With a probability of error*[4]

$$Prob_E \leq \frac{1}{N},$$

Grover's algorithm finds the unknown state $|x_0\rangle$ *at a computational cost of*

$$O\left(\sqrt{N}\lg N\right)$$

Proof.

Part 1. The probability of error $Prob_E$ of finding the hidden state $|x_0\rangle$ is given by

$$Prob_E = \cos^2\left[(2K+1)\beta\right] ,$$

where

$$\begin{cases} \beta &= \sin^{-1}\left(\frac{1}{\sqrt{N}}\right) \\ K &= round\left(\frac{\pi}{4\beta} - \frac{1}{2}\right) \end{cases},$$

[4]If the reader prefers to use the *floor* function rather than the *round* function, then probability of error becomes $Prob_E \leq \frac{4}{N} - \frac{4}{N^2}$.

where "*round*" is the function that rounds to the nearest integer. Hence,

$$\frac{\pi}{4\beta} - 1 \leq K \leq \frac{\pi}{4\beta} \implies \frac{\pi}{2} - \beta \leq (2K+1)\,\beta \leq \frac{\pi}{2} + \beta$$

$$\implies \sin\beta = \cos\left(\tfrac{\pi}{2} - \beta\right) \geq \cos\left[(2K+1)\,\beta\right] \geq \cos\left(\tfrac{\pi}{2} + \beta\right) = -\sin\beta$$

Thus,

$$Prob_E = \cos^2\left[(2K+1)\,\beta\right] \leq \sin^2\beta = \sin^2\left(\sin^{-1}\left(\frac{1}{\sqrt{N}}\right)\right) = \frac{1}{N}$$

Part 2. The computational cost of the Hadamard transform $H = \bigotimes_0^{n-1} H^{(2)}$ is $O(n) = O(\lg N)$ single qubit operations. The transformations $-I_{|0\rangle}$ and $I_{|x_0\rangle}$ each carry a computational cost of $O(1)$.

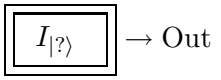
$\mathbb{STEP}$ 1 is the computationally dominant step. In $\mathbb{STEP}$ 1 there are $O\left(\sqrt{N}\right)$ iterations. In each iteration, the Hadamard transform is applied twice. The transformations $-I_{|0\rangle}$ and $I_{|x_0\rangle}$ are each applied once. Hence, each iteration comes with a computational cost of $O\left(\lg N\right)$, and so the total cost of $\mathbb{STEP}$ 1 is $O(\sqrt{N}\lg N)$.

□

6. An example of Grover's algorithm

As an example, we search a database consisting of $N = 2^n = 8$ records for an unknown record with the unknown label $x_0 = 5$. The calculations for this example were made with OpenQuacks, which is an open source quantum simulator Maple package developed at UMBC and publically available.

We are given a blackbox computing device



that implements as an oracle the unknown unitary transformation

$$I_{|x_0\rangle} = I_{|5\rangle} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We cannot open up the blackbox $\rightarrow \boxed{\boxed{I_{|?\rangle}}} \rightarrow$ to see what is inside. So we do not know what $I_{|x_0\rangle}$ and x_0 are. The only way that we can glean some information about x_0 is to apply some chosen state $|\psi\rangle$ as input, and then make use of the resulting output.

Using of the blackbox $\rightarrow \boxed{\boxed{I_{|?\rangle}}} \rightarrow$ as a component device, we construct a computing device $\rightarrow \boxed{\boxed{-HI_{|0\rangle}HI_{|?\rangle}}} \rightarrow$ which implements the unitary operator

$$Q = -HI_{|0\rangle}HI_{|x_0\rangle} = \frac{1}{4}\begin{pmatrix} -3 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -3 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -3 \end{pmatrix}$$

We do not know what unitary transformation Q is implemented by the device $\rightarrow \boxed{\boxed{-HI_{|0\rangle}HI_{|?\rangle}}} \rightarrow$ because the blackbox $\rightarrow \boxed{\boxed{I_{|?\rangle}}} \rightarrow$ is one of its essential components.

$\boxed{\mathbb{STEP}\ 0.}$ We begin by preparing the known state

$$\boxed{|\psi_0\rangle = H\,|0\rangle = \tfrac{1}{\sqrt{8}}\,(1,1,1,1,1,1,1,1)^{transpose}}$$

$\boxed{\mathbb{STEP}\ 1.}$ We proceed to loop

$$K = round\left(\frac{\pi}{4\sin^{-1}\left(1/\sqrt{8}\right)} - \frac{1}{2}\right) = 2$$

times in $\mathbb{STEP}$ 1.

ITERATION 1. On the first iteration, we obtain the unknown state

$$\boxed{|\psi_1\rangle = Q\,|\psi_0\rangle = \tfrac{1}{4\sqrt{2}}\,(1,1,1,1,5,1,1,1)^{transpose}}$$

ITERATION 2. On the second iteration, we obtain the unknown state

$$\boxed{|\psi_2\rangle = Q\,|\psi_1\rangle = \tfrac{1}{8\sqrt{2}}\,(-1,-1,-1,-1,11,-1,-1,-1)^{transpose}}$$

and branch to $\mathbb{STEP}$ 2.

$\boxed{\mathbb{STEP}\ 2.}$ We measure the unknown state $|\psi_2\rangle$ to obtain either

$$|5\rangle$$

with probability

$$Prob_{Success} = \sin^2\left((2K+1)\,\beta\right) = \frac{121}{128} = 0.9453$$

or some other state with probability

$$Prob_{Failure} = \cos^2\left((2K+1)\,\beta\right) = \frac{7}{128} = 0.0547$$

and then exit.

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