Introduction to Quantum Information Processing

Lecture 16

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Overview of Lecture 16

- The GHZ “paradox”
- The Bell inequality and its violation:
  - Physicist’s perspective
  - Computer Scientist’s perspective
- The magic square game

preliminaries

Quantum information can apparently be used to substantially reduce ***computation*** costs for a number of interesting problems

How does quantum information affect the ***communication costs*** of information processing tasks?

We explore this issue ...

Entanglement and signaling

Entangled states, such as $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$,

can be used to perform some intriguing feats, such as ***teleportation*** and ***superdense coding***

But they ***cannot*** be used to “signal instantaneously”

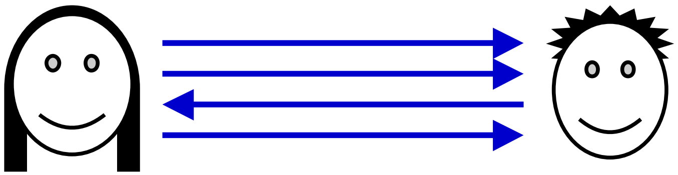
Any operation performed on one system has no affect on the state of the other system (its reduced density matrix)

Basic communication scenario

Goal: convey n bits from Alice to Bob

Basic communication scenario

Bit communication:

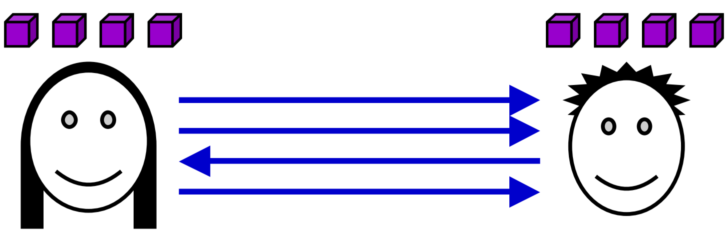


Cost: n

Qubit communication:

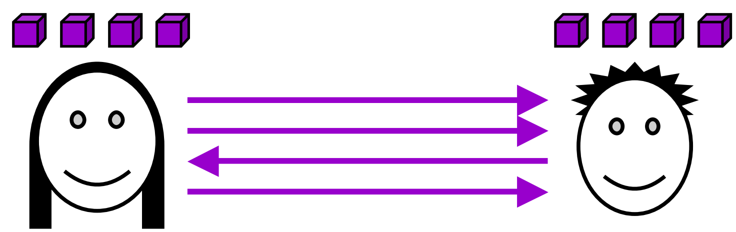
Cost: n [Holevo's Theorem, 1973]

Bit communication
& prior entanglement:



Cost: n (can be deduced)

Qubit communication
& prior entanglement:



Cost: $n/2$ superdense coding

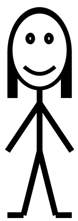
[Bennett & Wiesner, 1992]

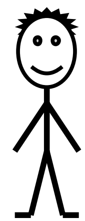
nonlocality a là GHZ

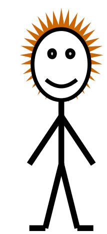
GHZ scenario

[Greenberger, Horne, Zeilinger, 1980]

Input: r s t







Output: $a \leftarrow r$ $b \leftarrow \neg s$ $c \leftarrow 1$

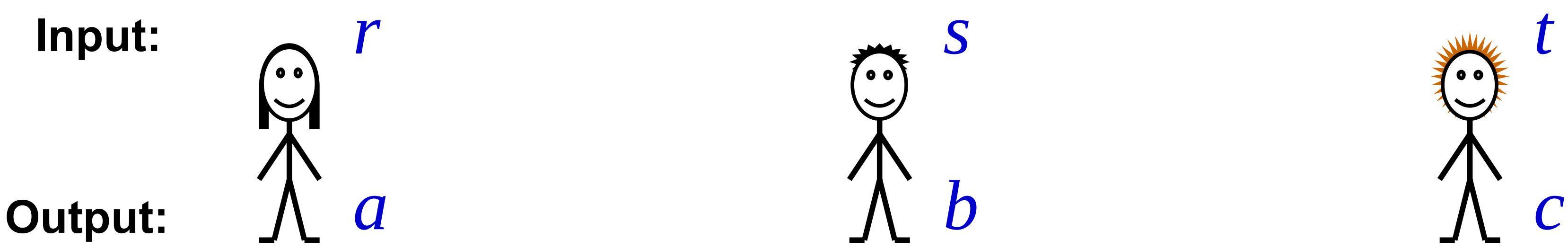
Alice **Bob** **Carol**

Rules of the game:

1. It is promised that $r \oplus s \oplus t = 0$
2. No communication after inputs received
3. They ***win*** if $a \oplus b \oplus c = r \vee s \vee t$

rst	$a \oplus b \oplus c$	abc
000	0	011
011	1	001
101	1	111
110	1	101

No perfect strategy for GHZ



rst	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

General deterministic strategy:

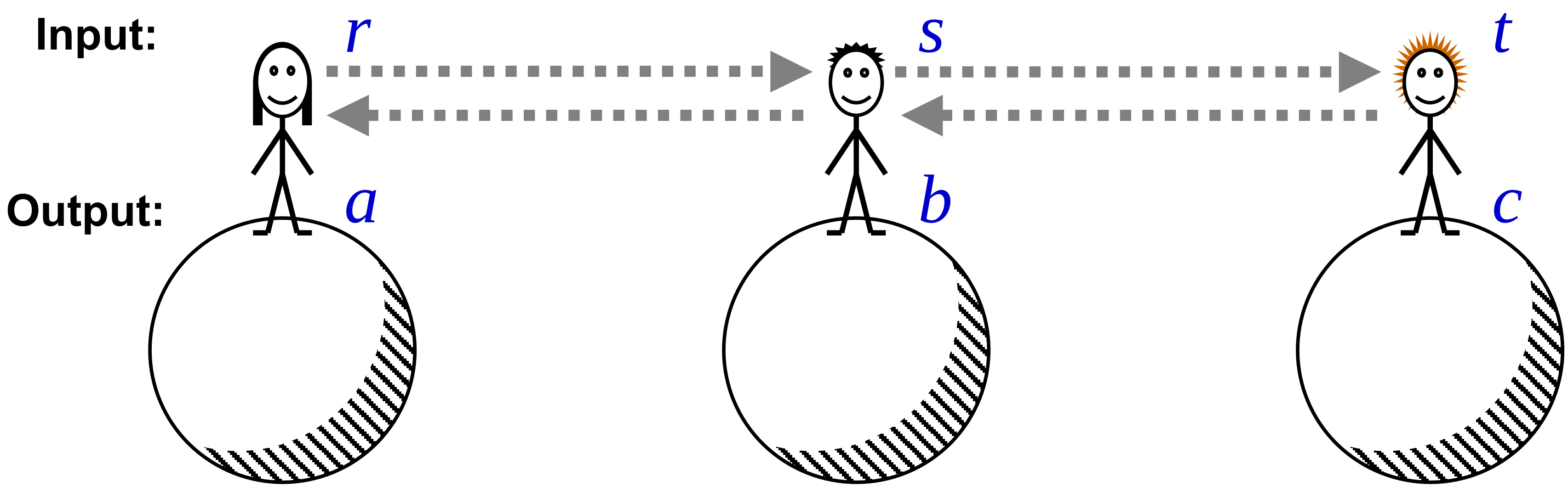
$a_0, a_1, b_0, b_1, c_0, c_1$

Winning conditions:

$$\begin{cases} a_0 \oplus b_0 \oplus c_0 = 0 \\ a_0 \oplus b_1 \oplus c_1 = 1 \\ a_1 \oplus b_0 \oplus c_1 = 1 \\ a_1 \oplus b_1 \oplus c_0 = 1 \end{cases}$$

Has no solution, thus no perfect strategy exists

GHZ: preventing communication

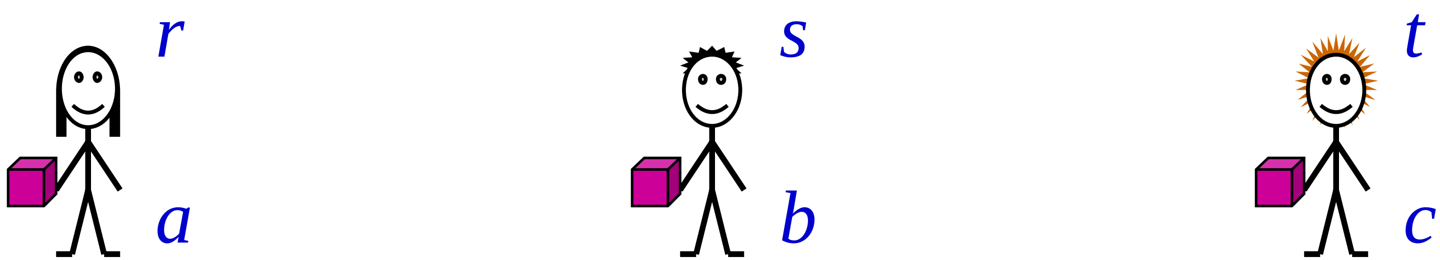


Input and output events can be ***space-like*** separated:
so signals at the speed of light are not fast enough for cheating

What if Alice, Bob, and Carol ***still*** keep on winning?

"GHZ Paradox" explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$



<u>Alice's strategy:</u>

1. if $r = 1$ then apply H to qubit
2. measure qubit and set a to result

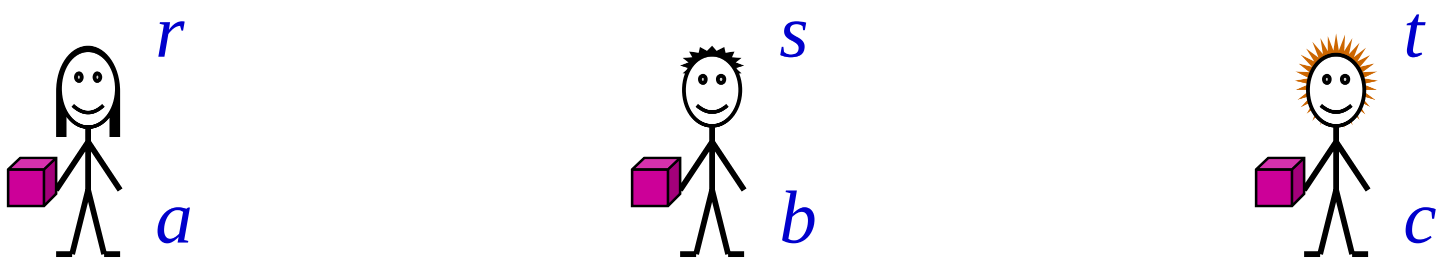
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

<u>Bob's & Carol's strategies:</u> similar

Case 1 ($rst = 000$)**:** state is measured directly …

“GHZ Paradox” explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$



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<u>Bob’s & Carol’s strategies:</u> similar

Case 2 ($rst = 011$)**:** new state $|001\rangle + |010\rangle - |100\rangle + |111\rangle$

Cases 3 & 4 ($rst = 101$ **&** 110)**:** similar by symmetry

GHZ: conclusions

- For the GHZ game, any ***classical*** team succeeds with probability at most ¾
- Allowing the players to communicate would enable them to succeed with probability 1
- Entanglement cannot be used to communicate
- Nevertheless, allowing the players to have entanglement enables them to succeed with probability 1
- Thus, entanglement is a useful resource for the task of ***winning the GHZ game***

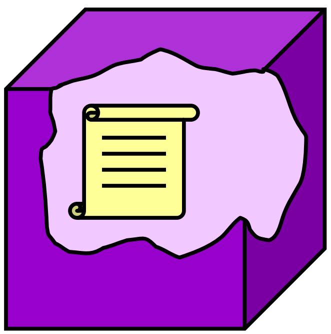
Bell’s Inequality and its violation part I

Bell’s Inequality and its violation

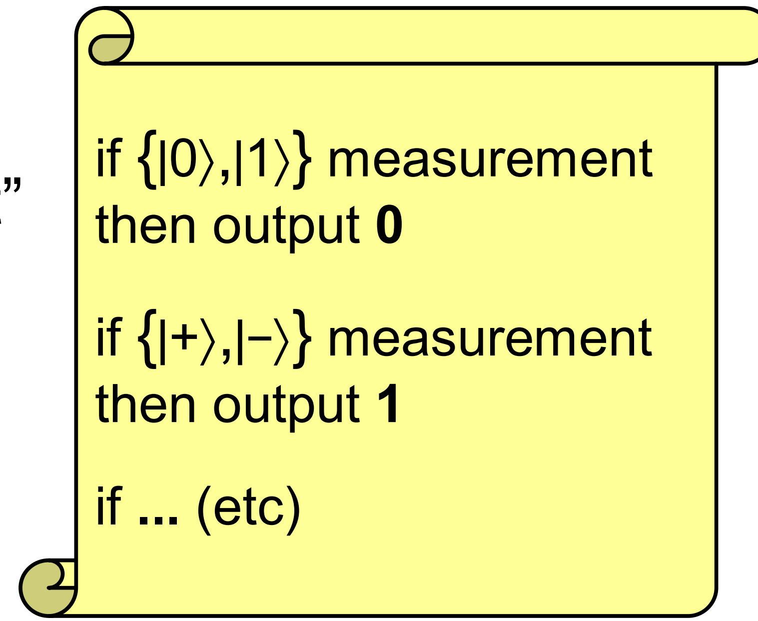
Part I: physicist’s view:

Can a quantum state have ***pre-determined*** outcomes for each possible measurement that can be applied to it?

qubit:



where the “manuscript” is something like this:

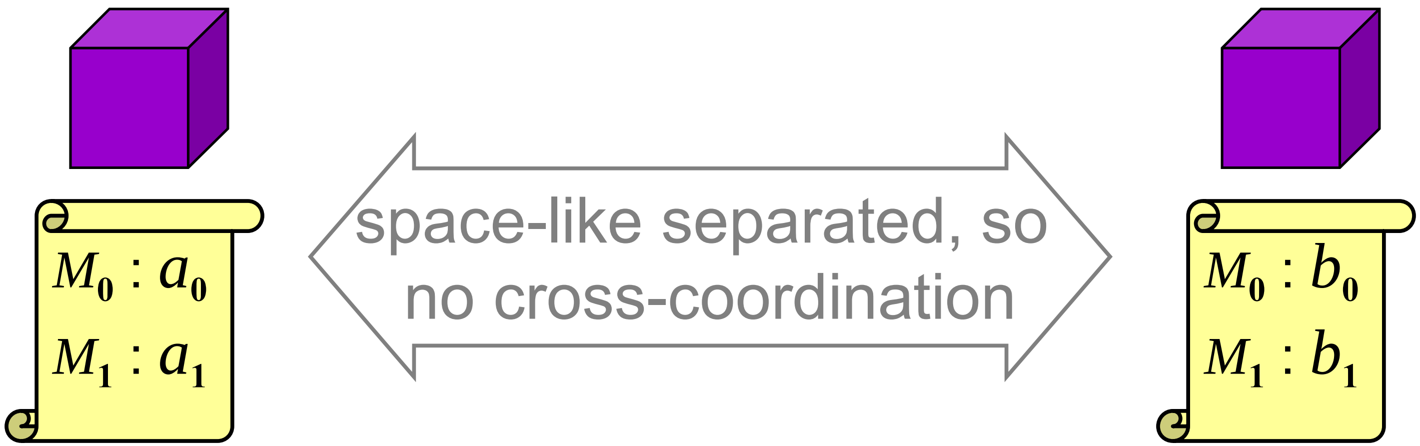


(called ***hidden variables***)

[Bell, 1964]
[Clauser, Horne, Shimony, Holt, 1969]

Bell Inequality

Imagine a two-qubit system, where one of two measurements, called M_0 and M_1, will be applied to each qubit:



Define: $A_0 = (-1)^{a_0}$ $A_1 = (-1)^{a_1}$ $B_0 = (-1)^{b_0}$ $B_1 = (-1)^{b_1}$

Claim: $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

Proof: $A_0 (B_0 + B_1) + A_1 (B_0 - B_1) \leq 2$

one is ± 2 and the other is 0

Bell Inequality

$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$ is called a ***Bell Inequality****

Question: could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?

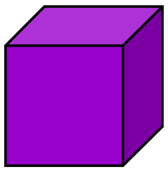
Answer 1: ***no, not directly***, because A_0, A_1, B_0, B_1 cannot all be measured (only ***one*** $A_s B_t$ term can be measured)

Answer 2: ***yes, indirectly***, by making many runs of this experiment: pick a random $st \in \{00, 01, 10, 11\}$ and then measure with M_s and M_t to get the value of $A_s B_t$

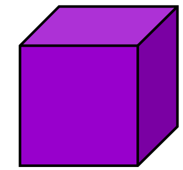
The ***average*** of $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ should be ≤ ½

* also called CHSH Inequality

Violating the Bell Inequality



Two-qubit system in state
$|\phi\rangle = |00\rangle - |11\rangle$



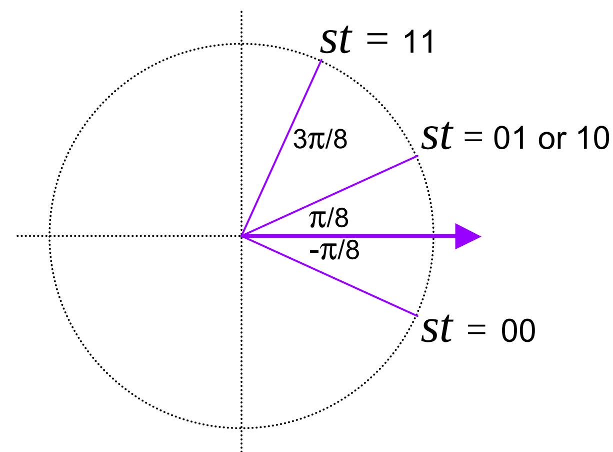
Applying rotations θ_A and θ_B yields:

$$\cos(\theta_A + \theta_B)\,\underbrace{(|00\rangle - |11\rangle)}_{AB\,=\,+1} + \sin(\theta_A + \theta_B)\,\underbrace{(|01\rangle + |10\rangle)}_{AB\,=\,-1}$$

Define

M_0: rotate by $-\pi/16$ then measure

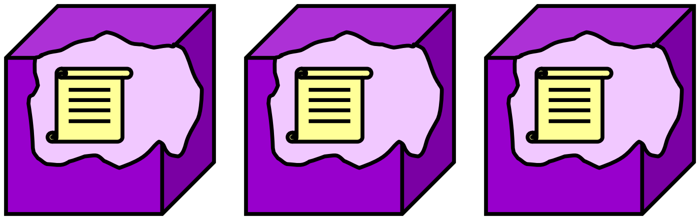
M_1: rotate by $+3\pi/16$ then measure



$\cos^2(\pi/8) = ½ + ¼\sqrt{2}$

Then $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ all have expected value $½\sqrt{2}$, which ***contradicts*** the upper bound of ½

Bell Inequality violation: summary



Assuming that quantum systems are governed by ***local hidden variables*** leads to the Bell inequality

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$$

But this is ***violated*** in the case of Bell states (by a factor of $\sqrt{2}$)

Therefore, no such hidden variables exist

This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted

Bell's Inequality and its violation part II

Bell's Inequality and its violation

Part II: computer scientist's view:

input: s t

output: a b

Rules: 1. No communication after inputs received

2. They ***win*** if $a \oplus b = s \wedge t$

st	$a \oplus b$
00	0
01	0
10	0
11	1

With classical resources, $\Pr[a \oplus b = s \wedge t] \leq 0.75$

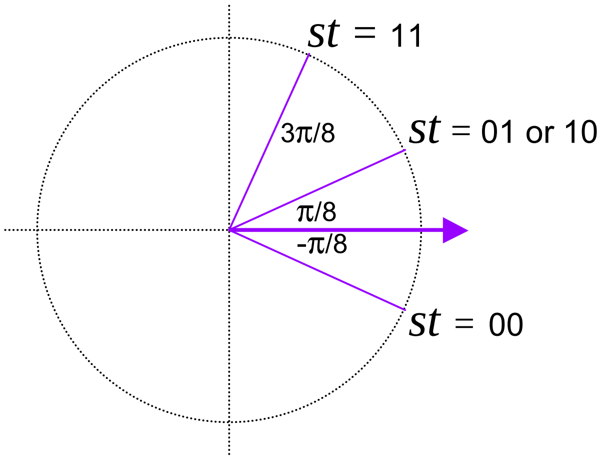
But, with prior entanglement state $|00\rangle - |11\rangle$,
$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = ½ + ¼\sqrt{2} = 0.853\ldots$

The quantum strategy

- Alice and Bob start with entanglement

$|\phi\rangle = |00\rangle - |11\rangle$

- **Alice:** if $s = 0$ then rotate by $\theta_A = -\pi/16$ else rotate by $\theta_A = +3\pi/16$ and measure



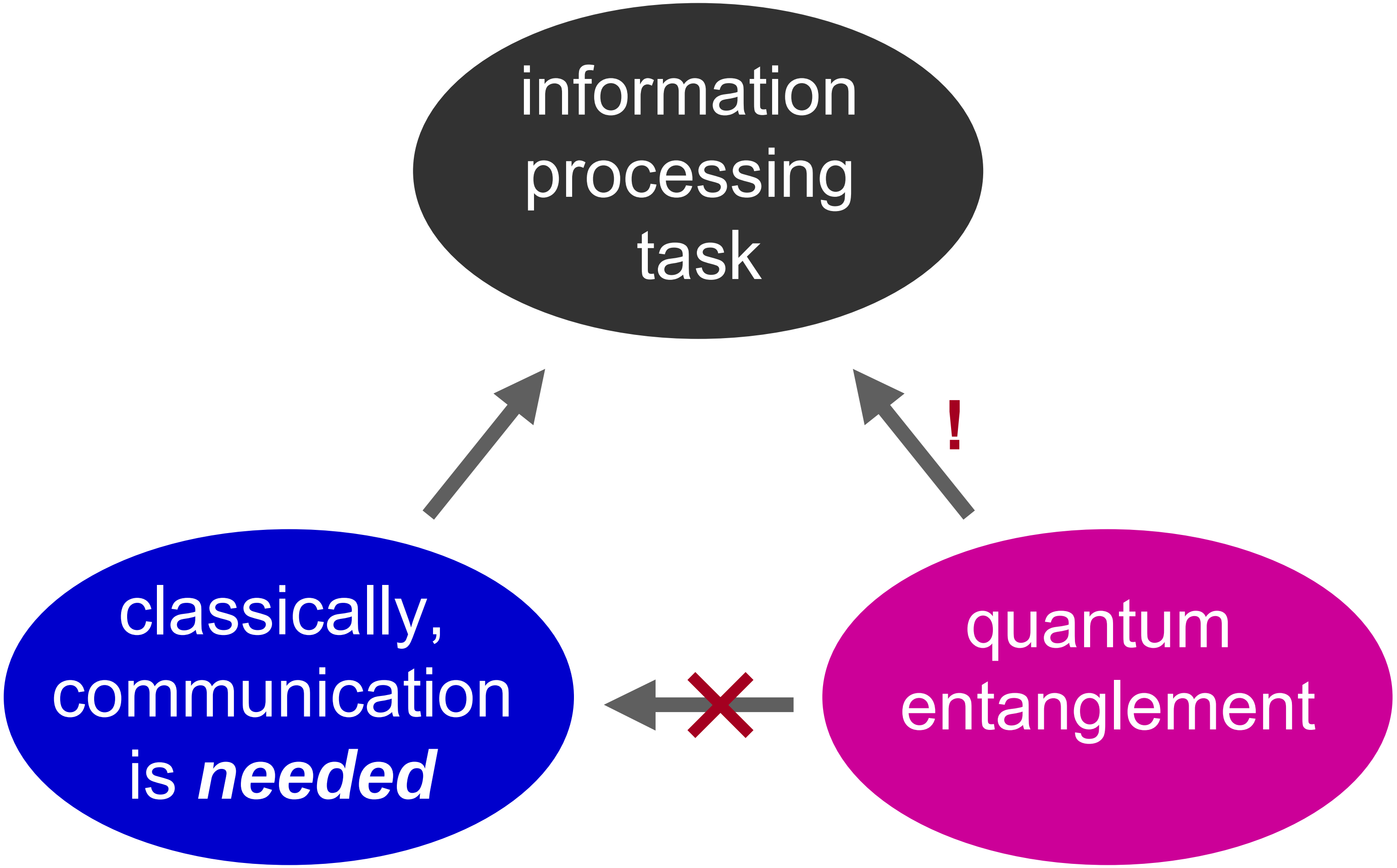
- **Bob:** if $t = 0$ then rotate by $\theta_B = -\pi/16$ else rotate by $\theta_B = +3\pi/16$ and measure

$\cos(\theta_A - \theta_B)\,(|00\rangle - |11\rangle) + \sin(\theta_A - \theta_B)\,(|01\rangle + |10\rangle)$

Success probability:

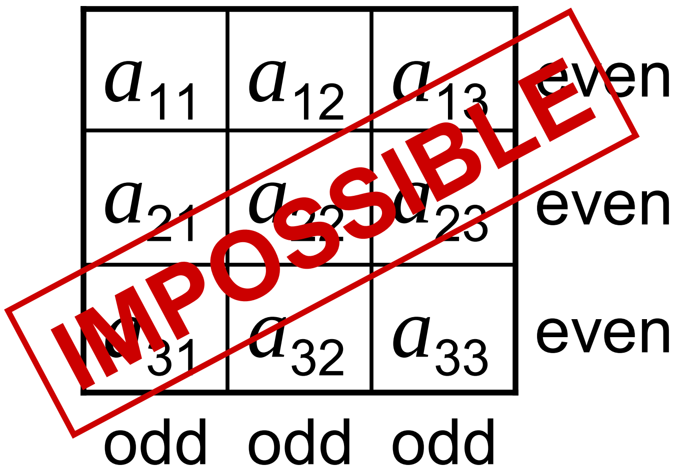
$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2} = 0.853\ldots$

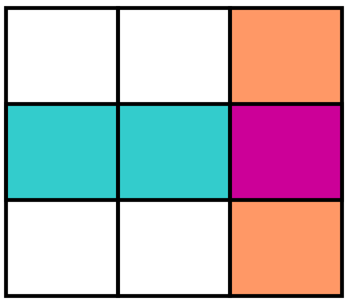
Nonlocality in operational terms



Magic square game

Problem: fill in the matrix with bits such that each row has even parity and each column has odd parity





Game: ask Alice to fill in one row and Bob to fill in one column

They ***win*** iff parities are correct and bits agree at intersection

Success probabilities: $8/9$ classical and 1 quantum

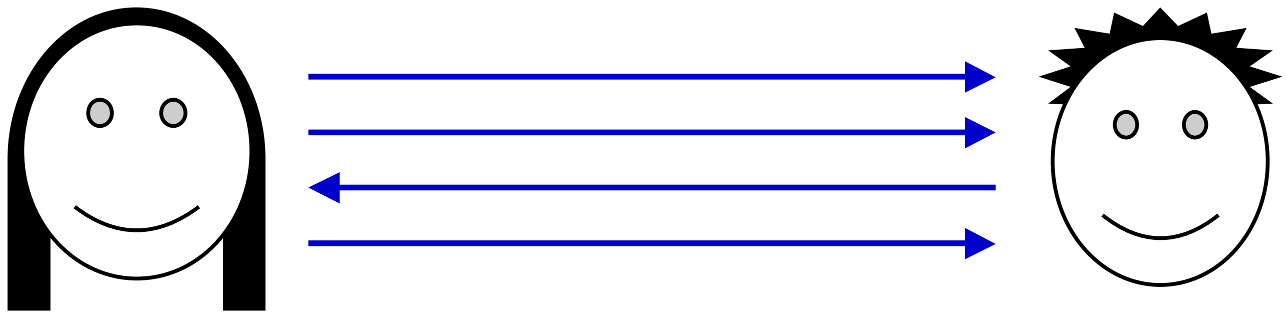
[Aravind, 2002] (details omitted here)

preview of communication complexity

Classical Communication Complexity

[Yao, 1979]

$x_1 x_2 \ldots x_n$ $y_1 y_2 \ldots y_n$



$f(x,y)$

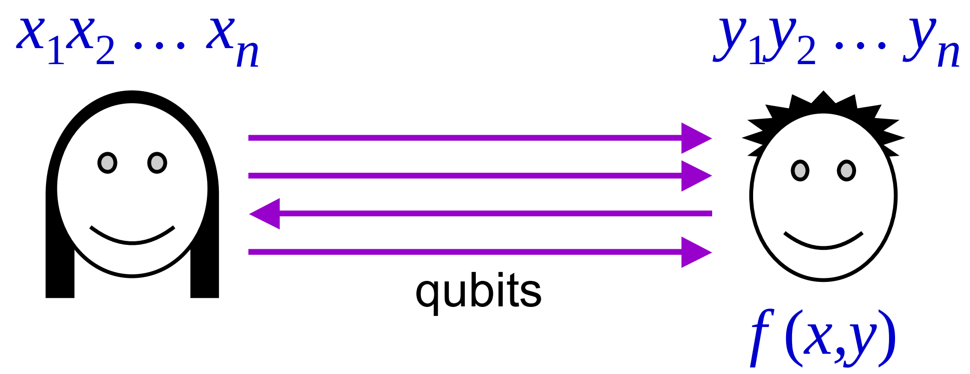
E.g. equality function: $f(x,y) = 1$ if $x = y$, and 0 if $x \neq y$

Any ***deterministic*** protocol requires n bits communication

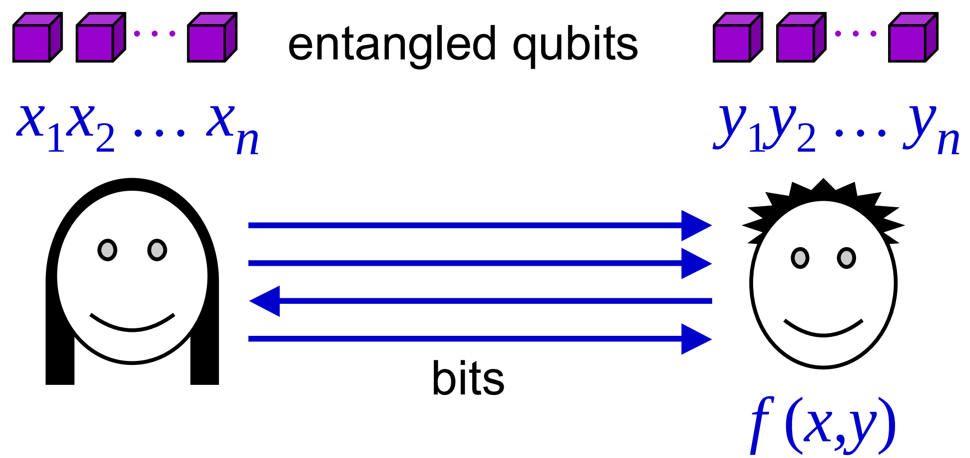
Probabilistic protocols can solve with only $O(\log(n/\varepsilon))$ bits communication (error probability ε)

Quantum Communication Complexity

Qubit communication



Prior entanglement



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