# Introduction to Quantum Information Processing

**Lecture 18**

**Richard Cleve**

# Overview of Lecture 18

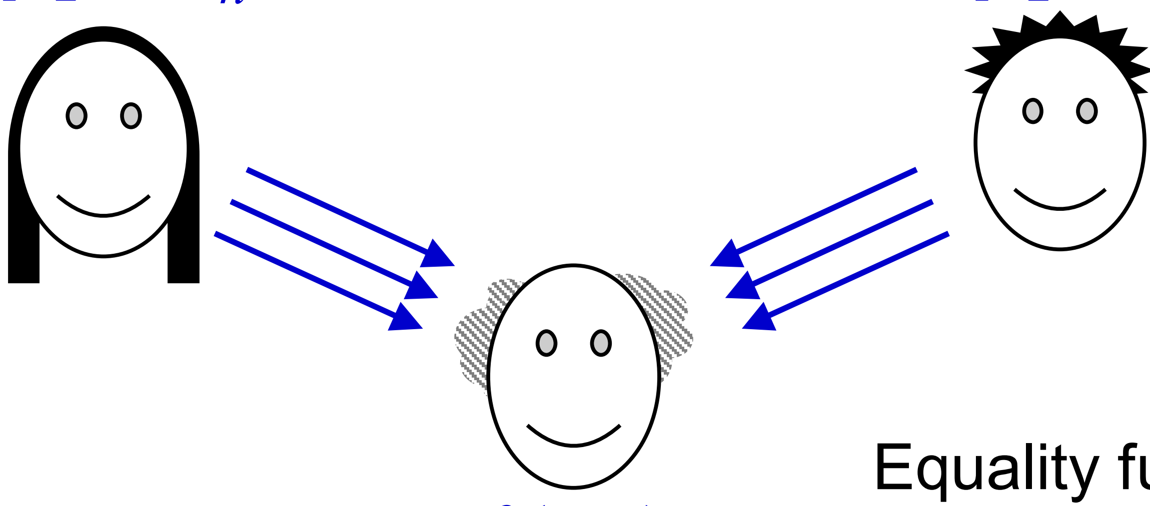
- Continuation of fingerprinting
- Hidden matching problem
- Restricted-equality nonlocality
- Universal sets of gates

# quantum fingerprints

# Equality revisited

## in simultaneous message model

$x_1 x_2 \ldots x_n$ $\qquad$ $y_1 y_2 \ldots y_n$



$f(x,y)$

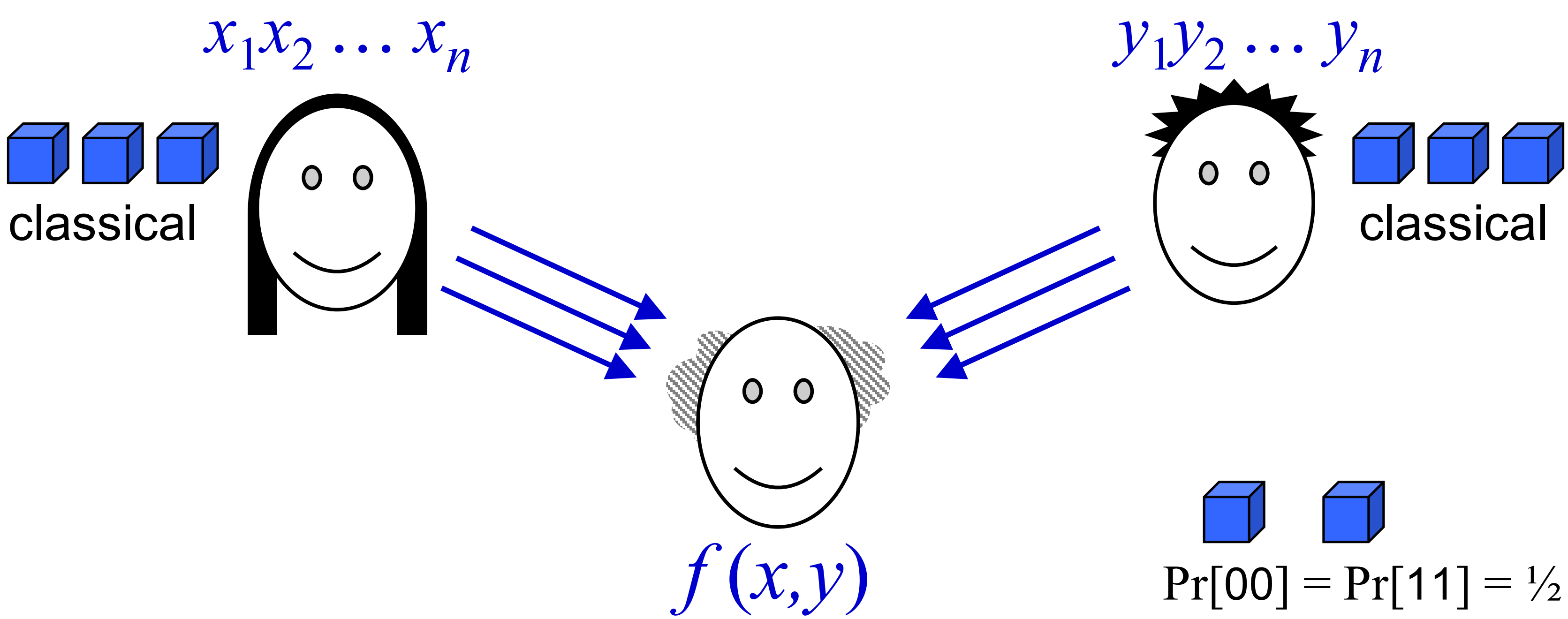
Equality function:

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

**Exact protocols:** require $2n$ bits communication

# Equality revisited

## in simultaneous message model



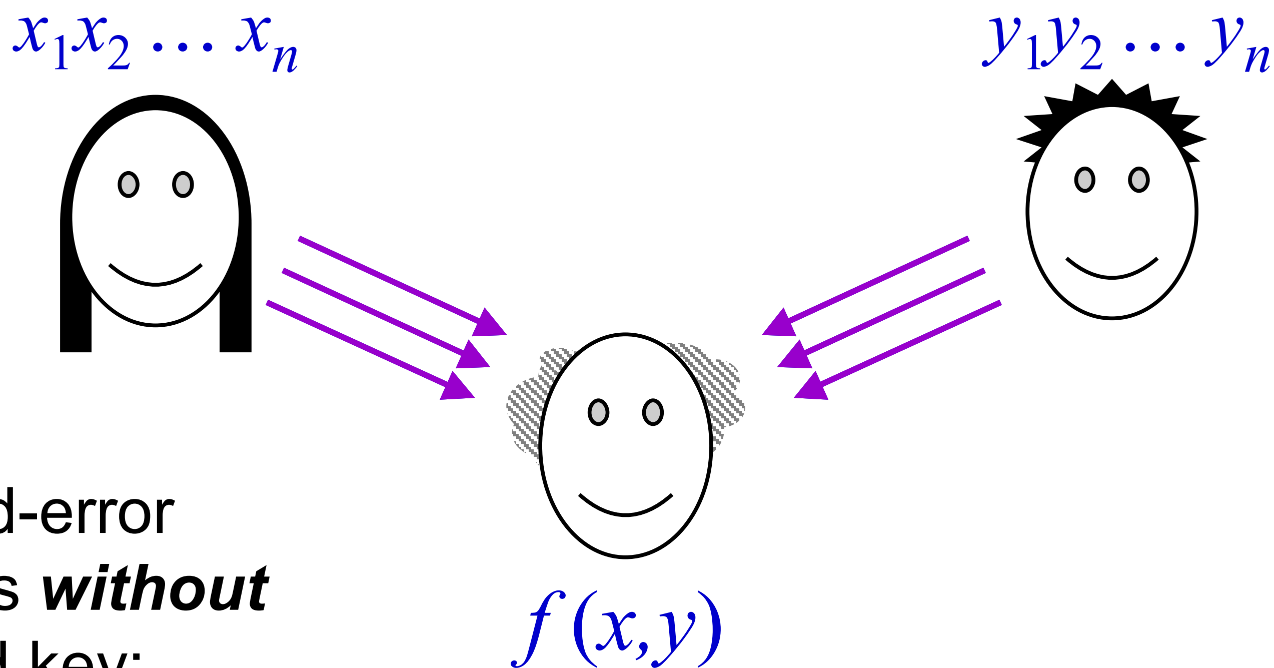

**Bounded-error protocols with a shared random key:** require only $O(1)$ bits communication

Error-correcting code: $e(x) =$ 1 0 1 1 1 1 0 1 0 1 1 0 0 1 1 0 0 1

$e(y) =$ 0 1 1 0 1 0 0 1 0 0 1 1 0 0 1 0 1 0

random $k$

# Equality revisited
## in simultaneous message model




Bounded-error protocols ***without*** a shared key:

**Classical:** $\theta(n^{1/2})$

**Quantum:** $\theta(\log n)$

[A ‘96] [NS ‘96] [BCWW ‘01]

# Quantum fingerprints

**Question 1:** how many orthogonal states in $m$ qubits?

**Answer:** $2^m$

Let $\varepsilon$ be an arbitrarily small positive constant

**Question 2:** how many ***almost orthogonal**** states in $m$ qubits?

(* where $|\langle\psi_x|\psi_y\rangle| \leq \varepsilon$ )

**Answer:** $2^{2^{am}}$, for some constant $a > 0$

**To be continued during next lecture …**

# Quantum fingerprints

**Question 1:** how many orthogonal states in $m$ qubits?

**Answer:** $2^m$

Let $\varepsilon$ be an arbitrarily small positive constant

**Question 2:** how many ***almost orthogonal**** states in $m$ qubits?

(***** where $|\langle\psi_x|\psi_y\rangle| \le \varepsilon$ )

**Answer:** $2^{2^{am}}$, for some constant $a > 0$

The states can be constructed via a suitable (classical) error-correcting code, which is a function $e:\{0,1\}^n \rightarrow \{0,1\}^{cn}$ where, for all $x \neq y$, $dcn \le \Delta(e(x),e(y)) \le (1-d)cn$ ($c$, $d$ are constants)

# Construction of *almost* orthogonal states

Set $|\psi_x\rangle = \frac{1}{\sqrt{cn}} \sum_{k=1}^{cn} (-1)^{e(x)_k} |k\rangle$ for each $x \in \{0,1\}^n$ ($\log(cn)$ qubits)

Then $\langle\psi_x|\psi_y\rangle = \frac{1}{cn} \sum_{k=1}^{cn} (-1)^{[e(x)\oplus e(y)]_k} |k\rangle = 1 - \frac{2\Delta(e(x),e(y))}{cn}$

Since $dcn \leq \Delta(e(x),e(y)) \leq (1-d)cn$, we have $|\langle\psi_x|\psi_y\rangle| \leq 1-2d$

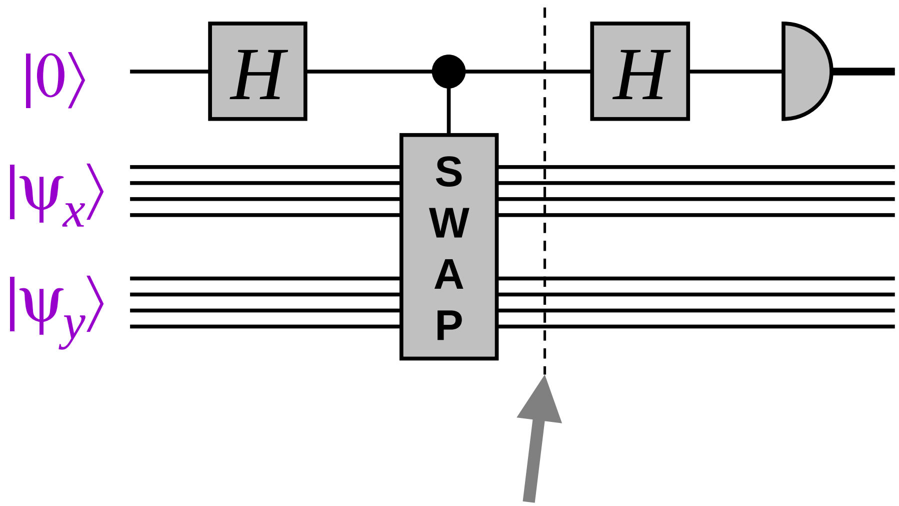
By duplicating each state, $|\psi_x\rangle \otimes |\psi_x\rangle \otimes \ldots \otimes |\psi_x\rangle$, the pairwise inner products can be made arbitrarily small: $(1-2d)^r \leq \varepsilon$

**Result:** $m = r\log(cn)$ qubits storing $2^n = 2^{(1/c)2^{m/r}}$ different states

# Quantum fingerprints

Let $|\psi_{000}\rangle, |\psi_{001}\rangle, \ldots, |\psi_{111}\rangle$ be $2^n$ states on $O(\log n)$ qubits such that $|\langle\psi_x|\psi_y\rangle| \leq \varepsilon$ for all $x \neq y$

Given $|\psi_x\rangle|\psi_y\rangle$, one can check if $x = y$ or $x \neq y$ as follows:



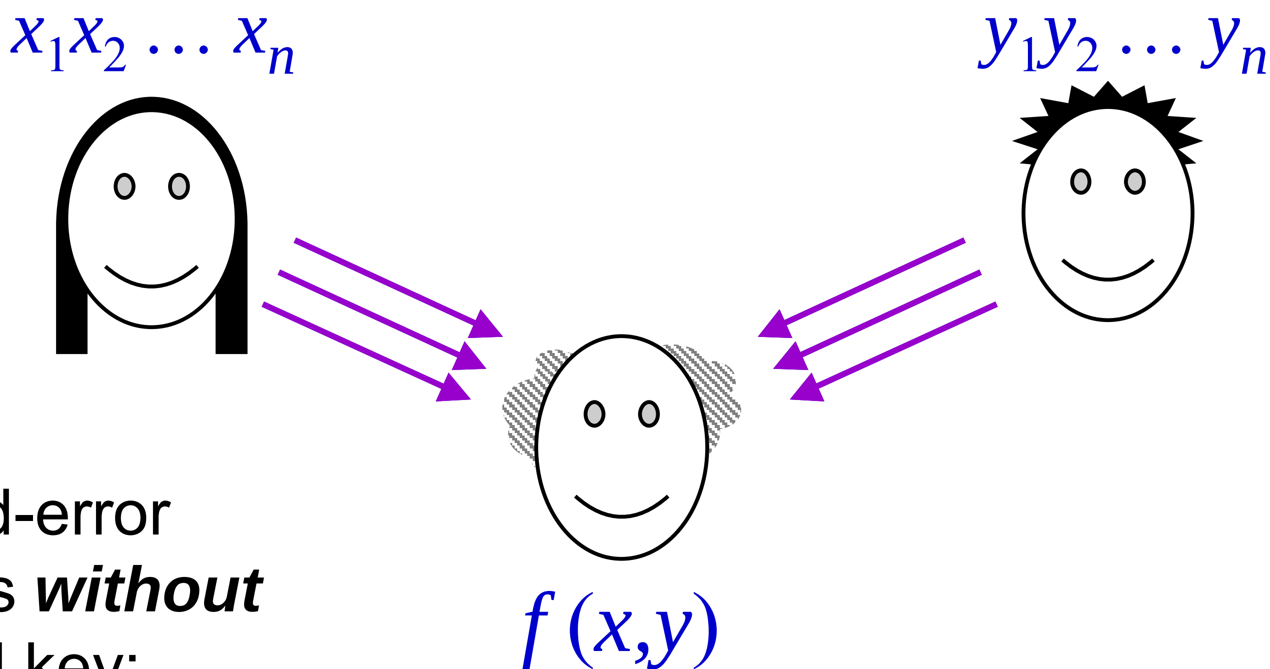

if $x = y$, $\Pr[\text{output} = 0] = 1$

if $x \neq y$, $\Pr[\text{output} = 0] = (1+\varepsilon^2)/2$

Intuition: $|0\rangle|\psi_x\rangle|\psi_y\rangle + |1\rangle|\psi_y\rangle|\psi_x\rangle$

**Note:** error probability can be reduced to $((1+\varepsilon^2)/2)^r$

# Equality revisited
## in simultaneous message model



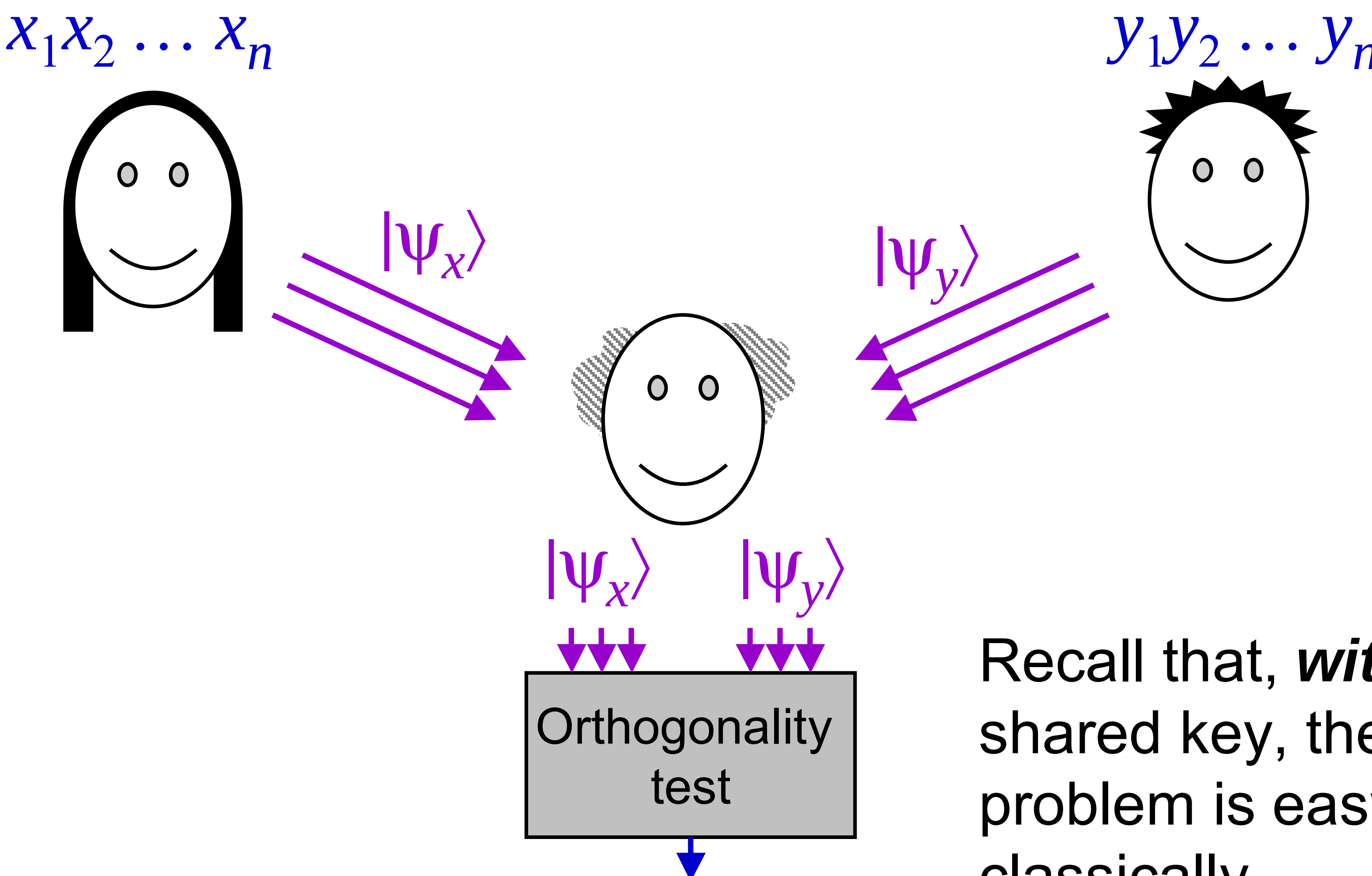

Bounded-error protocols ***without*** a shared key:

**Classical:** $\theta(n^{1/2})$

**Quantum:** $\theta(\log n)$

[A '96] [NS '96] [BCWW '01]

# Quantum protocol for equality
## in simultaneous message model



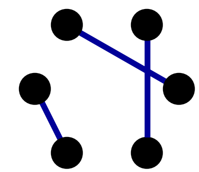

Recall that, ***with*** a shared key, the problem is easy classically ...

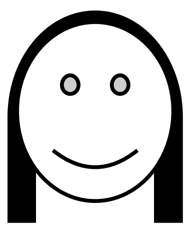
**... hidden matching problem**

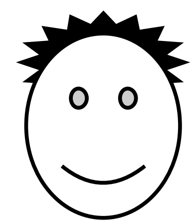
# Hidden matching problem

For this problem, a quantum protocol is exponentially more efficient than any classical protocol—even with a shared key

Inputs: $x \in \{0,1\}^n$

$M =$  ***matching*** on $\{1, 2, \ldots, n\}$



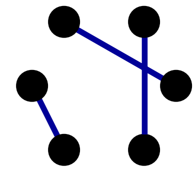


Output: $(i, j, x_i \oplus x_j)$, such that $(i, j) \in M$

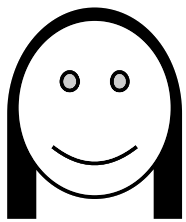
Only ***one-way*** communication (Alice to Bob) is permitted

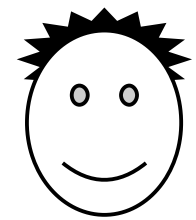
[Bar-Yossef, Jayram, Kerenidis, 2004]

# The hidden matching problem

$M =$  ***matching*** on $\{1,2, \ldots, n\}$

Inputs: $x \in \{0,1\}^n$





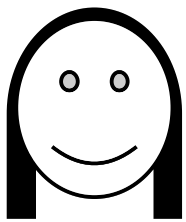
Output: $(i, j, x_i \oplus x_j)$, $(i, j) \in M$

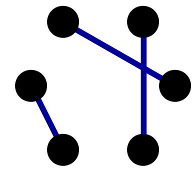
Classically, one-way communication is $\Omega(\sqrt{n})$, even with a shared classical key (the proof is omitted here)

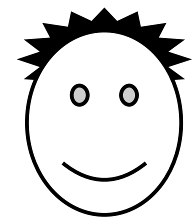
**Rough intuition:** Alice doesn't know which edges are in $M$, so she would have to send $\Omega(\sqrt{n})$ bits of the form $x_i \oplus x_j$ …

# The hidden matching problem

Inputs: $x \in \{0,1\}^n$



$M =$  ***matching*** on $\{1,2,\ldots,n\}$



Output: $(i, j, x_i \oplus x_j)$, $(i, j) \in M$

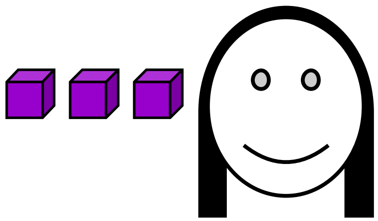
**Quantum protocol:** Alice sends $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(-1)^{x_k}|k\rangle$ ($\log n$ qubits)

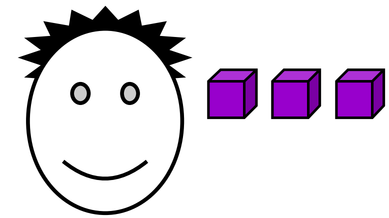
Bob measures in $|i\rangle \pm |j\rangle$ basis, $(i, j) \in M$, and uses the outcome's relative phase to determine $x_i \oplus x_j$

# nonlocality revisited

# Restricted-equality nonlocality

inputs: $x$ ($n$ bits) $y$ ($n$ bits)





outputs: $a$ ($\log n$ bits) $b$ ($\log n$ bits)

**Precondition:** either $x = y$ or $\Delta(x,y) = n/2$

**Required postcondition:** $a = b$ iff $x = y$

With classical resources, $\Omega(n)$ bits of communication needed ***for an exact solution****

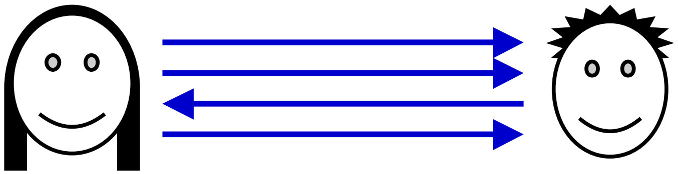
With $(|00\rangle + |11\rangle)^{\otimes \log n}$ prior entanglement, no communication is needed at all*

[BCT '99]

$*$ Technical details similar to restricted equality of Lecture 17

# Restricted-equality nonlocality

**Bit communication:**

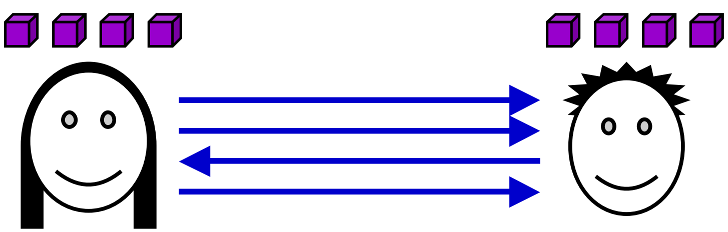


**Cost:** $\theta(n)$

**Qubit communication:**

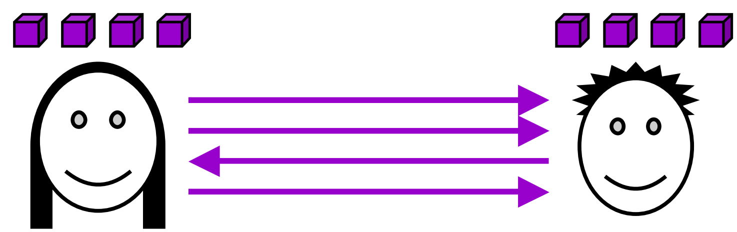
**Cost:** $\log n$

**Bit communication & prior entanglement:**



**Cost:** zero

**Qubit communication & prior entanglement:**



**Cost:** zero

# Nonlocality and communication complexity conclusions

- Quantum information affects communication complexity in interesting ways
- There is a rich interplay between quantum communication complexity and:
  - quantum algorithms
  - quantum information theory
  - other notions of complexity theory …

# universality of two-qubit gates

# A universal set of gates

**Theorem:** any unitary operation $U$ acting on $k$ qubits can be decomposed into $O(4^k)$ CNOT and one-qubit gates

(This was stated in Lecture 5 without a proof)

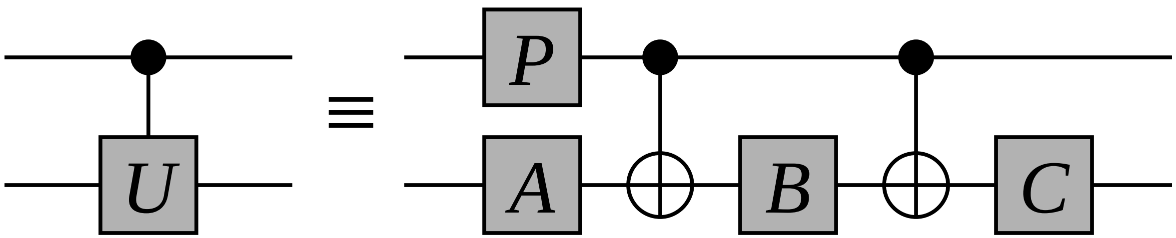
**Proof sketch** (for a slightly worse bound of $O(k^2 4^k)$) **:**
We first show how to simulate a controlled-$U$, for any one-qubit unitary $U$

**Fact:** for any one-qubit unitary $U$, there exist $A$, $B$, $C$, and λ, such that:

- $A\,B\,C = I$
- $e^{i\lambda}\,A\,X\,B\,X\,C = U$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

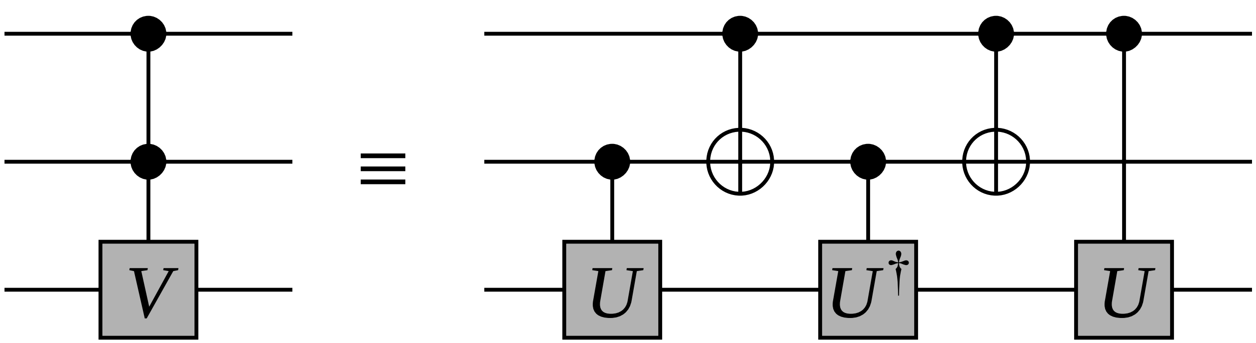
# A universal set of gates

The aforementioned fact implies




where $P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}$

Using such controlled-$U$ gates, one can simulate controlled-controlled-$V$ gates, for any unitary $V$, as follows:

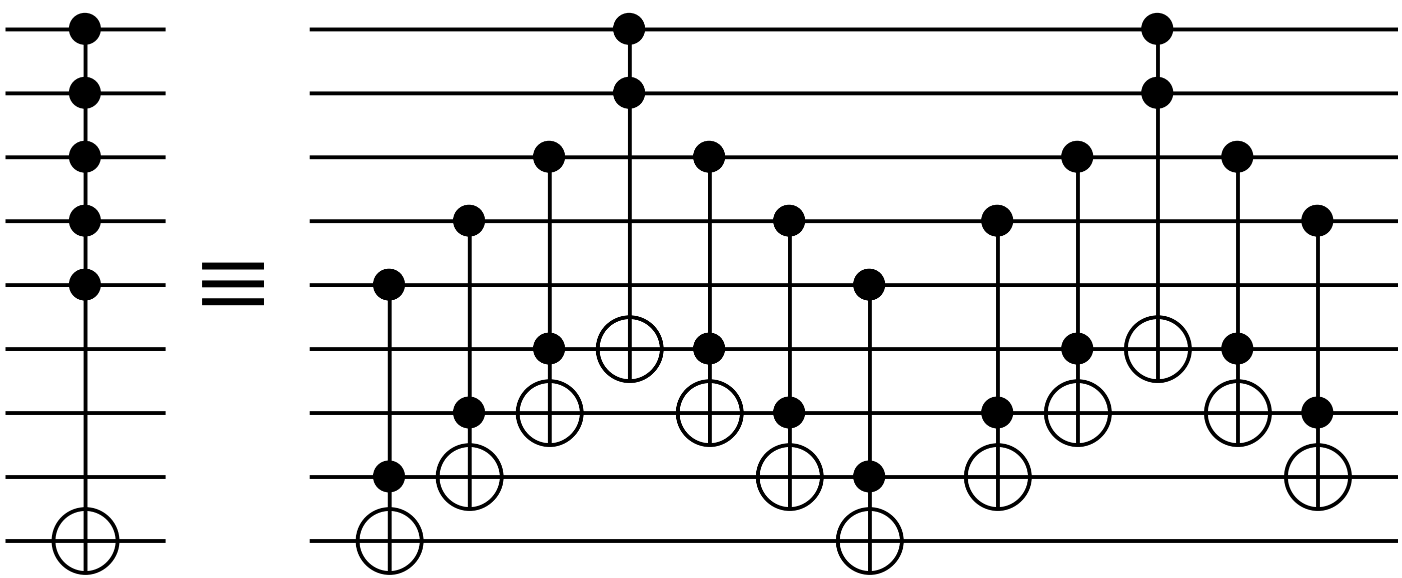



where $V = U^2$

# A universal set of gates

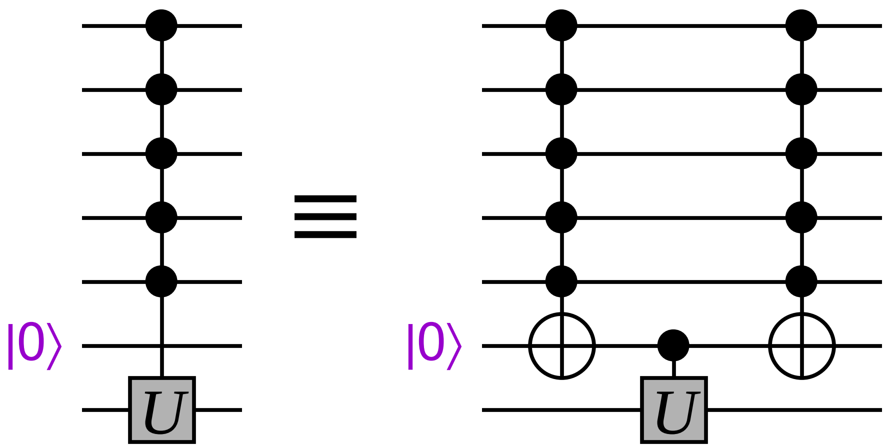
When $U = X$, this construction yields the 3-qubit ***Toffoli gate***

From this gate, ***generalized*** Toffoli gates can be constructed:



# A universal set of gates

From generalized Toffoli gates, ***generalized controlled-$U$*** gates (controlled-controlled- ... -$U$) can be constructed:




$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{10} & U_{11} \end{pmatrix}$$

# A universal set of gates

The approach essentially enables any $k$-qubit operation of the simple form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{00} & 0 & 0 & U_{01} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & U_{10} & 0 & 0 & U_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

to be computed with $O(k^2)$ CNOT and one-qubit gates

Any $2^k \times 2^k$ unitary matrix can be decomposed into a product of $O(4^k)$ such simple matrices

# A universal set of gates

**This completes the proof sketch**

Thus, the set of ***all*** one-qubit gates and the CNOT gate are ***universal*** in that they can simulate any other gate set

**Question:** is there a ***finite*** set of gates that is universal?

**Answer 1:** strictly speaking, ***no***, because this results in only countably many quantum circuits, whereas there are uncountably many unitary operations on $k$ qubits (for any $k$)

# THE END