communication tasks

How much classical information in *n* qubits?

2^n-1 complex numbers are needed to describe an arbitrary n-qubit pure quantum state:

$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \ldots + \alpha_{111}|111\rangle$

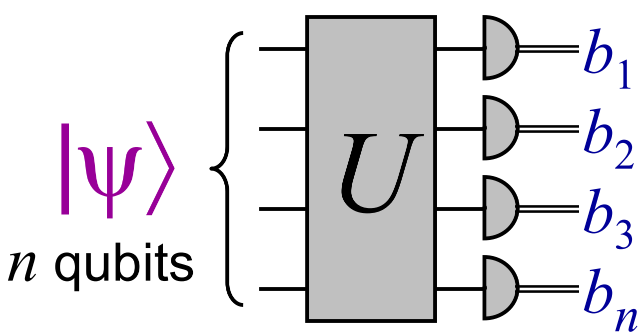
Does this mean that an exponential amount of classical information is stored in n qubits?

No! Holevo's Theorem [1973] implies: cannot convey more than n bits of information in n qubits

How much information does Nature have to store to maintain an n-qubit quantum state?

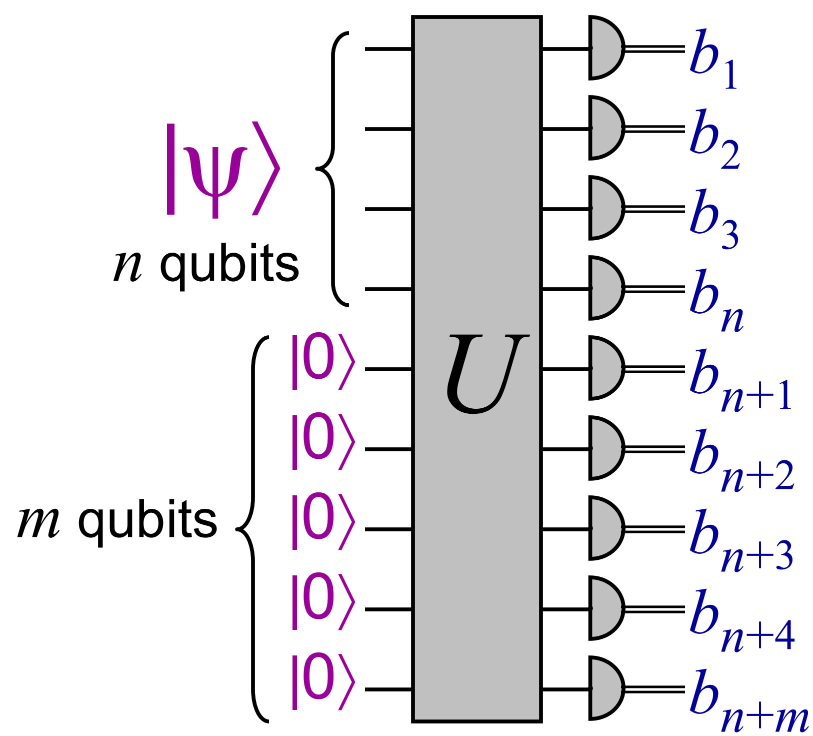
Holevo’s Theorem

Easy case:



$b_1 b_2 \ldots b_n$ cannot convey more than n bits!

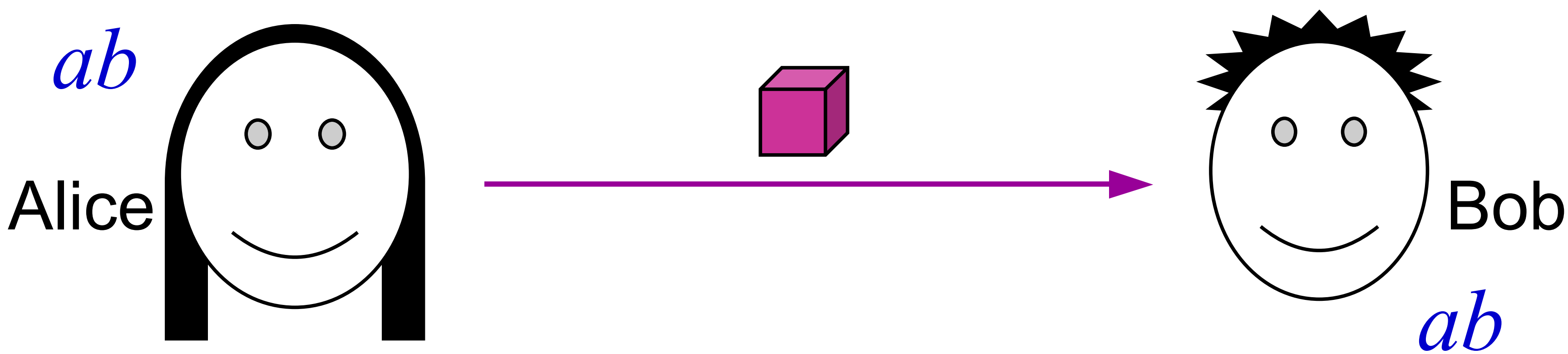
Hard case (the general case):



(proof is omitted here)

Superdense coding (prelude)

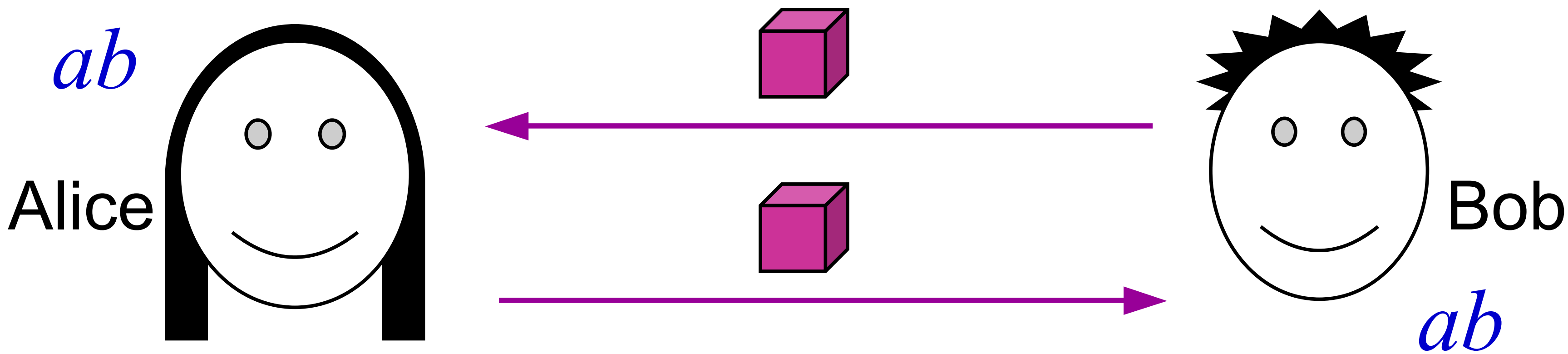
Suppose that Alice wants to convey two classical bits to Bob sending just one qubit



By Holevo's Theorem, this is ***impossible***

Superdense coding

In ***superdense coding***, Bob can send a qubit to Alice first



How can this help?

How superdense coding works

1. Bob creates the state $|00\rangle + |11\rangle$ and sends the ***first*** qubit to Alice

2. Alice: if $a = 1$ then apply X to qubit
   if $b = 1$ then apply Z to qubit
   send the qubit back to Bob

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

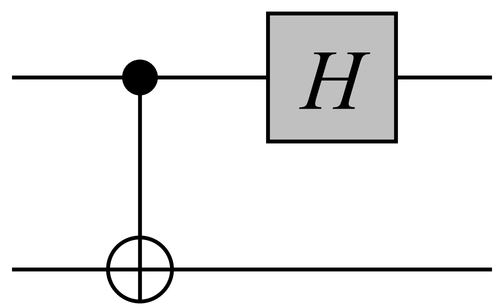
ab	state
00	$\|00\rangle + \|11\rangle$
01	$\|00\rangle - \|11\rangle$
10	$\|01\rangle + \|10\rangle$
11	$\|01\rangle - \|10\rangle$

} Bell basis

3. Bob measures the two qubits in the ***Bell basis***

Measurement in the Bell basis

Specifically, Bob applies



input	output
$\|00\rangle + \|11\rangle$	$\|00\rangle$
$\|00\rangle - \|11\rangle$	$\|01\rangle$
$\|01\rangle + \|10\rangle$	$\|10\rangle$
$\|01\rangle - \|10\rangle$	$\|11\rangle$

to his two qubits ...

and then measures them, yielding ab

This concludes superdense coding

Review of partial measurements

Suppose one measures just the ***first*** qubit of the state

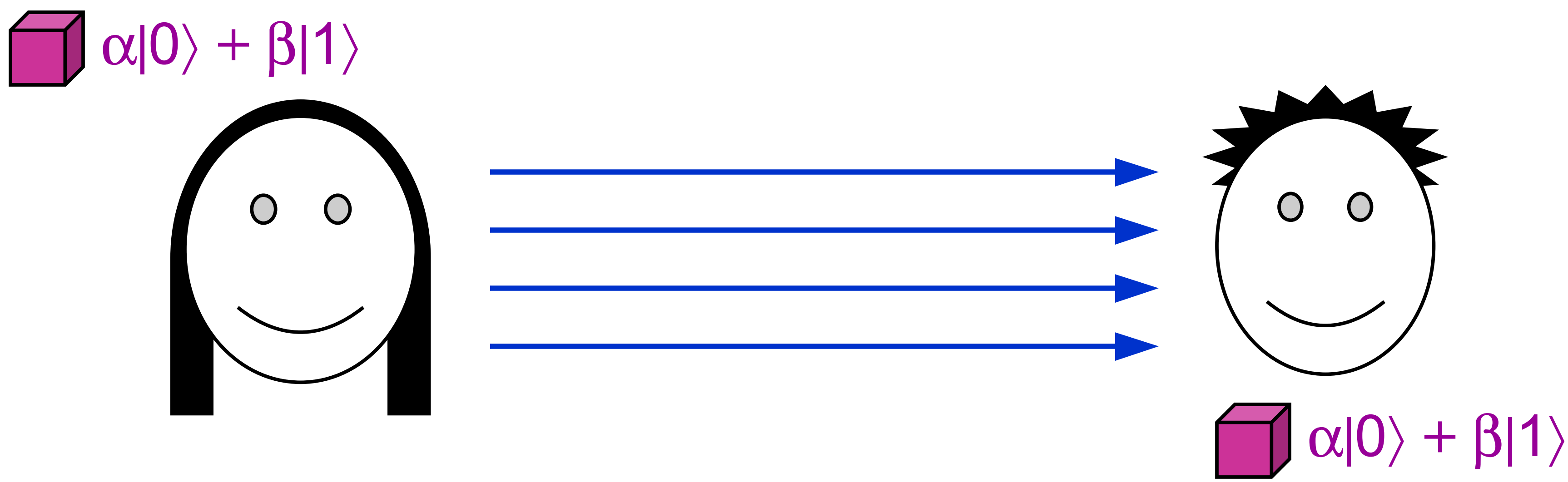
$$\frac{1}{2}|00\rangle + \frac{i}{\sqrt{3}}|01\rangle + \sqrt{\frac{5}{12}}|11\rangle = \sqrt{\frac{7}{12}}|0\rangle\left(\sqrt{\frac{3}{7}}|0\rangle + i\sqrt{\frac{4}{7}}|1\rangle\right) + \sqrt{\frac{5}{12}}|1\rangle|1\rangle$$

What is the result?

$$\begin{cases} 0, \quad \sqrt{\frac{3}{7}}|0\rangle + i\sqrt{\frac{4}{7}}|1\rangle & \text{with prob. } 7/12 \\ 1, \quad |1\rangle & \text{with prob. } 5/12 \end{cases}$$

Teleportation (prelude)

Suppose Alice wishes to convey a qubit to Bob by sending just classical bits

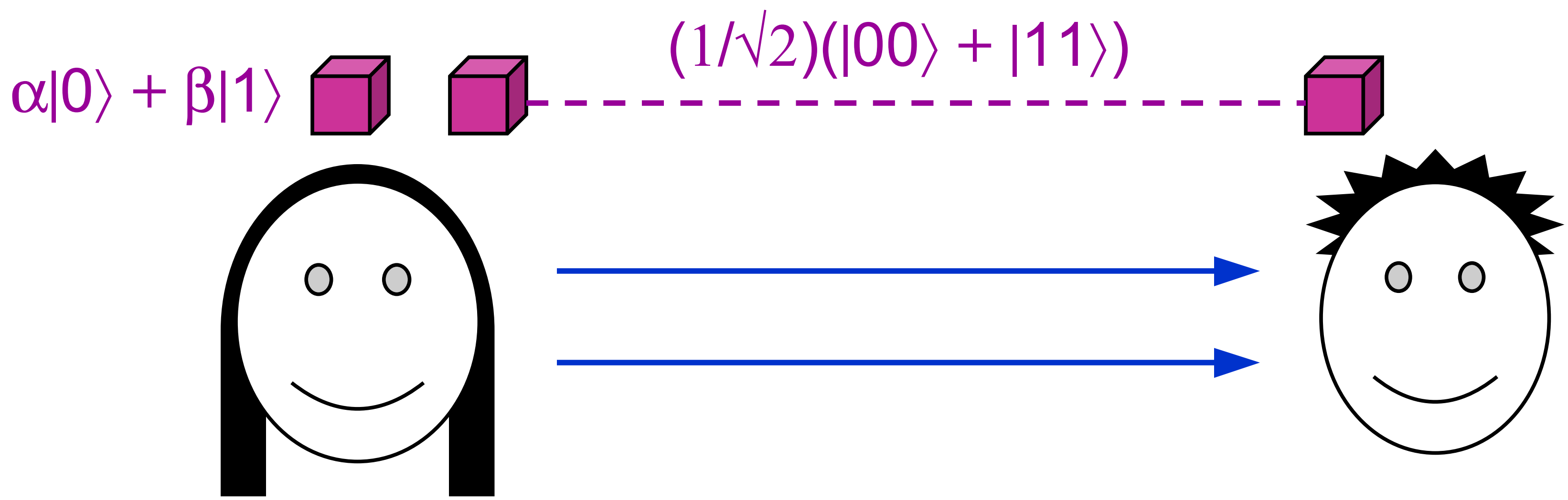


If Alice knows α and β, she can send approximations of them —but this requires infinitely many bits for perfect precision

Moreover, if Alice does ***not*** know α or β, she can at best acquire ***one bit*** about them by a measurement

Teleportation scenario

In teleportation, Alice and Bob also start with a Bell state



and Alice can send two classical bits to Bob

Note that the initial state of the three qubit system is:

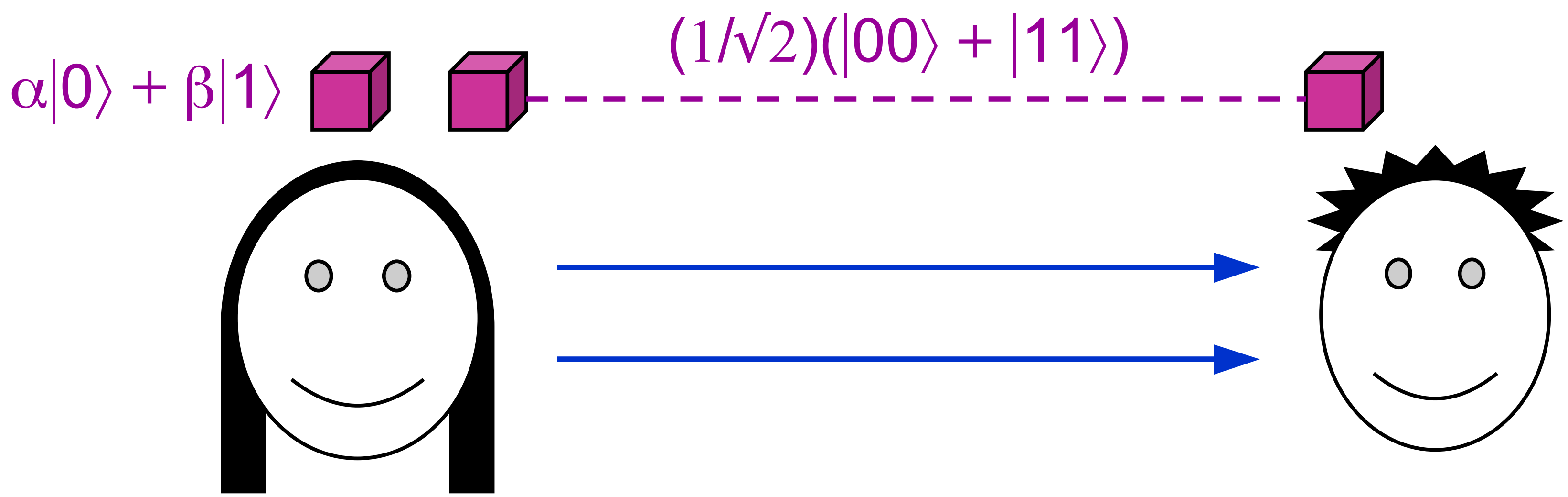
$(1/\sqrt{2})(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle)$

$= (1/\sqrt{2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

Recap of teleportation scenario

Goal: for Alice to convey her qubit to Bob

Resources: an entangled state and two bits communication



Note that the initial state of the three qubit system is:

$$(1/\sqrt{2})(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle)$$

$$= (1/\sqrt{2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

How teleportation works

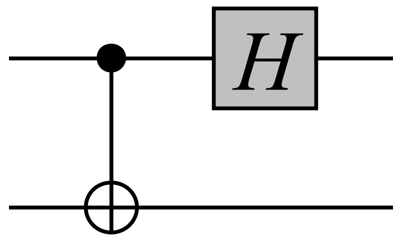
Initial state: $(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle)$ (omitting the $1/\sqrt{2}$ factor)

$$= \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle$$

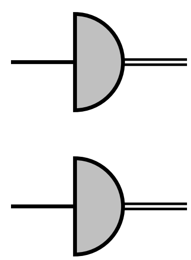
$$= \tfrac{1}{2}(|00\rangle + |11\rangle)(\alpha|0\rangle + \beta|1\rangle)$$
$$+ \tfrac{1}{2}(|00\rangle - |11\rangle)(\alpha|1\rangle + \beta|0\rangle)$$
$$+ \tfrac{1}{2}(|01\rangle + |10\rangle)(\alpha|0\rangle - \beta|1\rangle)$$
$$+ \tfrac{1}{2}(|01\rangle - |10\rangle)(\alpha|1\rangle - \beta|0\rangle)$$

Protocol: Alice measures her two qubits ***in the Bell basis*** and sends the result to Bob (who then “corrects” his state)

What Alice does specifically

Alice applies  to her two qubits, yielding:

$$\begin{cases} \tfrac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) \\ + \tfrac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) \\ + \tfrac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) \\ + \tfrac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle) \end{cases}$$



$$\begin{cases} (00,\ \alpha|0\rangle + \beta|1\rangle) & \text{with prob. } \tfrac{1}{4} \\ (01,\ \alpha|1\rangle + \beta|0\rangle) & \text{with prob. } \tfrac{1}{4} \\ (10,\ \alpha|0\rangle - \beta|1\rangle) & \text{with prob. } \tfrac{1}{4} \\ (11,\ \alpha|1\rangle - \beta|0\rangle) & \text{with prob. } \tfrac{1}{4} \end{cases}$$

Then Alice sends her two classical bits to Bob, who then adjusts his qubit to be $\alpha|0\rangle + \beta|1\rangle$ whatever case occurs

Bob's adjustment procedure

Bob receives two classical bits a, b from Alice, and:

if $b = 1$ he applies X to qubit

if $a = 1$ he applies Z to qubit

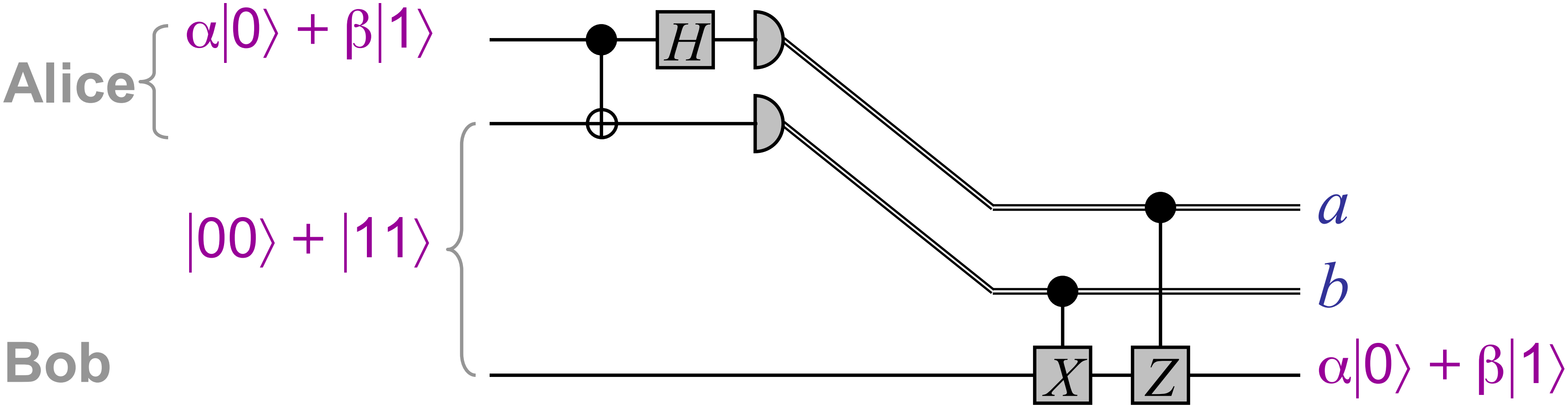
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

yielding:

$$\begin{cases} 00, & \alpha|0\rangle + \beta|1\rangle \\ 01, & X(\alpha|1\rangle + \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \\ 10, & Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle \\ 11, & ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle \end{cases}$$

Note that Bob acquires the correct state in each case

Summary of teleportation



Suggested exercise: try to work through the analysis of the teleportation protocol on your own