Moments and Deviations

497 - Randomized Algorithms

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1 Some More Probability

- If X and Y are *independent* then

  $$E[XY] = E[X]E[Y].$$

  Let $\mu_X = E[X]$.

- $\mathrm{var}[X] = E[(X-\mu_X)^2] = E[X^2] - \mu_X^2$ - this is the *variance*. Intuitively, this tells us how concentrated is the distribution of X.

- $\sigma_X = \sqrt{\mathrm{var}[X]}$: *standard deviation*. (i.e., variance is inconvenient to work with because it is a squared quantity).

- $\mathrm{var}[cX] = c^2 \,\mathrm{var}[X]$.

- For X,Y independent variables, we have $\mathrm{var}[X+Y] = \mathrm{var}[X] + \mathrm{var}[Y]$.

- *Bernoulli distribution*

  We flip a coin, get 1 (heads) with probability p, and 0 (i.e., tail) with probability $q = 1-p$. Let X be this random variable. The variable X is has *Bernoulli distribution with parameter* p. Then $E[X] = p$, and $\mathrm{var}[X] = pq$.

- *Binomial distribution*

  Assume that we repeat a Bernoulli experiments n times (independently!). Let $X_1, \ldots, X_n$ be the resulting random variables, and let $X = X_1 + \cdots + X_n$. The variable X has the *binomial distribution* with parameters n and p. We denote this fact by $X \sim B(n,p)$. We have

  $$b(k;n,p) = \Pr[X=k] = \binom{n}{k} p^k q^{n-k}.$$

  Also, $E[X] = np$, and $\mathrm{var}[X] = npq$.

2 Occupancy Problems (RA 3.1)

Problem 2.1 We are throwing m balls into n bins randomly (i.e., for every ball we randomly and uniformly pick a bin from the n available bins, and place the ball in the bin picked). What is the maximum number of balls in any bin? What is the number of bins which are empty? How many balls do we have to throw, such that all the bins are non-empty, with reasonable probability?

Observation 2.2 *Let $C_1, \ldots, C_n$ be random events (not necessarily independent). Than* $\Pr\left[\bigcup_{i=1}^{n} C_i\right] \leq \sum_{i=1}^{n} \Pr\left[C_i\right]$. *If $C_1, \ldots, C_n$ are* disjoint *then* $\Pr\left[\bigcup_{i=1}^{n} C_i\right] = \sum_{i=1}^{n} \Pr\left[C_i\right]$

Let X_i be the number of balls in the i-th bins, when we throw n balls into n bins (i.e., $m = n$). Clearly,

$$E\left[X_i\right] = \sum_{j=1}^{n} \Pr\left[\text{The } j\text{-th ball fall in } i\text{-th bin}\right] = n \cdot \frac{1}{n} = 1.$$

The probability that the first bin has exactly i balls is

$$\binom{n}{i}\left(\frac{1}{n}\right)^i\left(1-\frac{1}{n}\right)^{n-i} \leq \binom{n}{i}\left(\frac{1}{n}\right)^i \leq \left(\frac{ne}{i}\right)^i\left(\frac{1}{n}\right)^i = \left(\frac{e}{i}\right)^i$$

This follows by $\boxed{\text{For any } k \leq n, \text{ we have: } \left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.}$

Anyway, let $C_j(k)$ be the event that the j-th bin has k or more balls in it. Then,

$$\Pr\left[C_1(k)\right] \leq \sum_{i=k}^{n}\left(\frac{e}{i}\right)^i \leq \left(\frac{e}{k}\right)^k\left(1+\frac{e}{k}+\frac{e^2}{k^2}+\ldots\right) = \left(\frac{e}{k}\right)^k \frac{1}{1-e/k}.$$

Let $k^* = \lceil (3\ln n)/\ln\ln n \rceil$. Then,

$$\begin{aligned}
\Pr\left[C_1\left(k^*\right)\right] &\leq \left(\frac{e}{k^*}\right)^{k^*} \frac{1}{1-e/k^*} \leq 2\left(\frac{e}{(3\ln n)/\ln\ln n}\right)^{k^*} = 2\left(e^{1-\ln 3-\ln\ln n+\ln\ln\ln n}\right)^{k^*} \\
&\leq 2\left(e^{-\ln\ln n+\ln\ln\ln n}\right)^{k^*} \leq 2\exp\left(-3\ln n + 6\ln n\frac{\ln\ln\ln n}{\ln\ln n}\right) \leq 2\exp\left(-2.5\ln n\right) \leq \frac{1}{n^2},
\end{aligned}$$

for n large enough.

$$\Pr\left[\text{any bin contains more than } k \text{ balls}\right] \leq \sum_{i=1}^{n} C_i\left(k^*\right) \leq \frac{1}{n}.$$

Theorem 2.3 *With probability at least $1 - 1/n$, no bin has more than $k^* = \lceil (3\ln n)/\ln\ln n \rceil$ balls in it.*

Exercise 2.4 *Show that for $m = n\ln n$, with probability $1 - o(1)$, every bin has $O(\log n)$ balls.*

Probability of all bins to have exactly one ball

Next, we are interested in the probability that all m balls fall in distinct bins. Let X_i be the event that the i-th ball fell in a distinct bin from the first $i-1$ balls. We have:

$$\begin{aligned}\Pr\left[\bigcap_{i=2}^{m} X_i\right] &= \Pr\left[X_2\right]\prod_{i=3}^{m}\Pr\left[X_i \,\middle|\, \bigcap_{j=2}^{i-1} X_j\right] \leq \prod_{i=2}^{m}\left(\frac{n-i+1}{n}\right) \leq \prod_{i=2}^{m}\left(1-\frac{i-1}{n}\right) \leq \prod_{i=2}^{m} e^{-(i-1)/n} \\ &\leq \exp\left(-\frac{m(m-1)}{2n}\right),\end{aligned}$$

thus for $m = \left\lceil \sqrt{2n}+1 \right\rceil$, the probability that all the m balls fall in different bins is smaller than $1/e$.

The Markov and Chebyshev inequalities (RA 3.2)

Definition 2.5 For a random variable X assuming real values, its *expectation* is

$$E\left[Y\right] = \sum_y y \cdot \Pr\left[Y = y\right].$$

Similarly, for a function $f(\cdot)$, we have

$$E\left[f(Y)\right] = \sum_y f(y) \cdot \Pr\left[Y = y\right].$$

Theorem 2.6 (Markov Inequality) *Let Y be a random variable assuming only non-negative values. Then for all $t > 0$, we have*

$$\Pr\left[Y \geq t\right] \leq \frac{E\left[Y\right]}{t}$$

Proof:

$$E\left[Y\right] = \sum_{y\geq t} y\Pr\left[Y=y\right] + \sum_{y<t} y\Pr\left[Y=y\right] \geq \sum_{y\geq t} y\Pr\left[Y=y\right] \geq \sum_{y\geq t} t\Pr\left[Y=y\right] = t\Pr\left[Y\geq t\right]. \quad \blacksquare$$

Markov inequality is tight, to see that:

Exercise 2.7 *Define a random positive variable X, such that* $\Pr\left[X \geq kE\left[X\right]\right] = \frac{1}{k}$.

Definition 2.8 For a random variable X, with expectation μ_X, its *variance* is $\sigma_X^2 = E\left[(X-\mu_X)^2\right]$. The *standard deviation* is σ_X.

Theorem 2.9 (Chebychev inequality) $\Pr\left[|X-\mu_X| \geq t\sigma_X\right] \leq \dfrac{1}{t^2}$.

Proof: Note that

$$\Pr\left[|X-\mu_X| \geq t\sigma_X\right] = \Pr\left[(X-\mu_X)^2 \geq t^2\sigma_X^2\right].$$

Set $Y = (X-\mu_X)^2$. Clearly, $E\left[Y\right] = \sigma_X^2$. Now, apply Markov inequality to Y. $\blacksquare$