18.419 Random Walks and Polynomial-Time Algorithms	18 April, 2002

Lecture 17 : Ball Walk I

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In today's lecture, we will discuss ball-walk one of the random walks proposed for sampling in continuous spaces.

First for some notation. Let B_r denote the r-dimensional unit ball centered at the origin. If $r = n$, then we will drop the subscript r. Let δB denote the ball of radius δ centered at the origin. For any $u \in \mathbb{R}^n$, let $u + \delta B$ denote the translated ball δB such that its center is at u.

Let K be a continuous convex body. The ball-walk in K can now be described as follows:

Ball Walk

At any point $x \in K$,

1. Choose y uniformly at random from $x + \delta B$.
2. If $y \in K$, go to y, else stay at x.

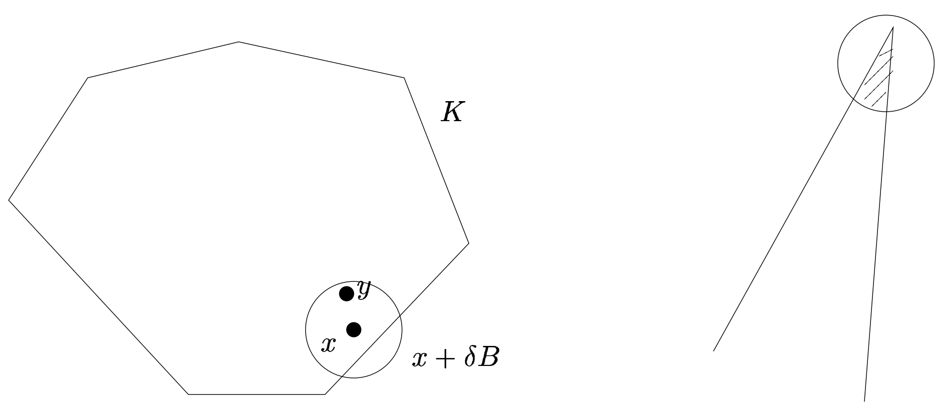

Figure 1: (a) Ball Walk (b) Points with low local conductance.

The ball-walk is smoother than the discrete-sampling methods proposed a couple of lectures back. However, akin to isolated points in the discrete methods, we have that at certain points (close to the boundary of K) there is a high probability of staying at the same point (See Figure 1(b)). This is captured by the fact that the local conductance $l(u)$[1] is low.

[1] Recall local conductance at $u \in K$ is defined as $P_u(K \backslash \{u\})$.

We can now describe the ball-walk in the language of Markov schemes. The probability density function is given as follows. For any $u \in K$,

$$\begin{aligned} p(u,v) &= \begin{cases} \frac{1}{\mathrm{Vol}(\delta B)} & \text{if } v \in K \backslash \{u\} \text{ and } |u-v| \le \delta \\ 0 & \text{otherwise} \end{cases} \\ P_u(A) &= \int_v p(u,v)dv \\ &= \int_{v \in (u+\delta B)\cap K} \frac{1}{\mathrm{Vol}(\delta B)} dv \end{aligned}$$

We can now easily check that the uniform distribution is a stationary distribution for the ball-walk. For the uniform distribution Q, $Q(A) = \mathrm{Vol}(A)/\mathrm{Vol}(K)$ and hence, $dQ(u) = du/\mathrm{Vol}(K)$. Thus,

$$\begin{aligned} \int_{u \in K} P_u(A)dQ(u) &= \int_{u \in K} \int_{v \in (u+\delta B)\cap K} \frac{1}{\mathrm{Vol}(\delta B)} dv dQ(u) \\ &= \frac{1}{\mathrm{Vol}(\delta B)} \int_{v \in K} \int_{u \in v+\delta B} dQ(u)dv \\ &= \frac{1}{\mathrm{Vol}(\delta B)} \cdot \frac{\mathrm{Vol}(A)}{\mathrm{Vol}(K)} \cdot \mathrm{Vol}(\delta B) \\ &= Q(A) \end{aligned}$$

The mixing time of a continuous random walk is related to the conductance[2] of the walk as indicated in the following theorem.

Theorem 1 (Lovász and Simonovits (1993)). *If the Markov chain is started in distribution Q_0, then for all $A \in \mathcal{A}$,*

$$|Q_t(A) - Q_0(A)| \le \sqrt{M}\left(1 - \frac{\Phi^2}{2}\right)^t$$

where $M = \sup_A \frac{Q_0(A)}{Q(A)}$.

Thus, to show that the ball-walk mixes well, we need to demonstrate that the conductance of the ball-walk is large. So, our goal for the rest of today's lecture and the next lecture will be to show that that Φ for the ball walk is large.

To begin with, we observe that the conductance can be no larger than the minimum local conductance. It is possible for certain points to have very low local conductance, however

[2]Conductance Φ is defined as $\Phi = \min_{0 < Q(A) \le \frac{1}{2}} \frac{\Phi(A)}{Q(A)}$ where $\Phi(A) = \int_{u \in A} P_u(\bar{A})dQ(u)$

the set of such points may have measure zero and hence is irrelevant to the ball-walk. Let $H_t = \{x \in K | l(x) \le t\}$. For any such H_t such that $Q(H_t) > 0$, we have $\Phi(H_t) \le tQ(H_t)$. Hence, $\Phi \le \Phi(H_t)/Q(t) \le t$. Thus, $\Phi \le l$.

We observe that only points close to the boundary can have low local conductance. For interior points, if $u + \delta B \subseteq K$, then $l(u) = 1$. For starters, we will assume that the local conductance of all points is bounded below by l. (i.e., $l(u) \ge l, \forall u \in K$.) We need to show that for all subsets $S \subseteq K$, $\Phi(S) = \int_S P_u(\bar{S}) dQ(u)$ is not too small compared to $Q(S)$. For this purpose, we shall show that if two points $x, y \in K$ are close, then the variation distance between the distribution attained after a random step from x and y is also small.

Suppose $x, y \in K$ such that $|x - y| = \delta$, then the volume of the intersection of the balls $(x + \delta B)$ and $(y + \delta B)$ can be computed as follows (See Figure 2):

$$\frac{\mathrm{Vol}\Big((x+\delta B) \cap (y + \delta B)\Big)}{\mathrm{Vol}(\delta B)} \le \frac{\mathrm{Vol}_{n-1}\left(\frac{\sqrt{3}}{2}\delta B_{n-1}\right) \cdot \delta}{\mathrm{Vol}(\delta B)} \le \sqrt{n} \cdot \left(\frac{\sqrt{3}}{2}\right)^{n-1}$$

Thus, if $|x-y| = \delta$, then the intersection is exponentially small. How small should $|x-y|$ be

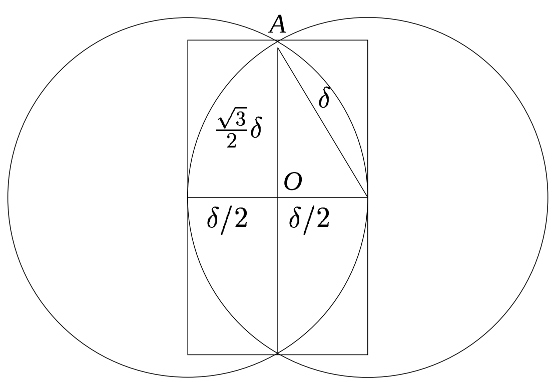

Figure 2: Intersection of $x + \delta B$ and $y + \delta B$ when $|x - y| = \delta$

such that the intersection is sufficiently large. For the intersection to be at least a constant fraction of $\mathrm{Vol}(\delta B)$, we need AO (see Figure 2) to be at least $(1 - 1/n)\delta$. In that case we have

$$\frac{1}{2}|x - y| \le \sqrt{\delta^2 - \left(1 - \frac{1}{n}\right)^2 \delta^2} \le \delta\sqrt{\frac{2}{n}}$$

Thus, we need to have x and y such that $|x - y|$ is at most $\delta\sqrt{8/n}$. Let us now compute

Figure 3: Intersection of $x + \delta B$ and $y + \delta B$ when $|x - y| \leq \frac{t\delta}{\sqrt{n}}$

the volume of the intersection when $|x - y| \leq t\delta/\sqrt{n}$ for some constant t. We need to compute the volume of the intersection as shown in Figure 3(a). But this volume is exactly the same as the volume of the ball excluding the middle strip in Figure 3(b). Hence, when $|x - y| \leq t\delta/\sqrt{n}$, we have

$$\begin{aligned}
\text{Vol}\Big((x + \delta B) \cap (y + \delta B)\Big) &= \text{Vol}(\delta B) - \text{Vol(Middle Strip in ball in Figure 3(b))} \\
&\geq \text{Vol}(\delta B) - \text{Vol}_{n-1}(\delta B_{n-1}) \cdot \frac{t\delta}{\sqrt{n}} \\
&\geq \text{Vol}(\delta B) - \frac{c\sqrt{n}}{\delta}\text{Vol}(\delta B)\frac{t\delta}{\sqrt{n}} \text{for some constant } c \\
&= (1 - ct)\text{Vol}(\delta B)
\end{aligned}$$

The last but one step follows from the fact that $\text{Vol}_n(\delta B_n) = 2\pi^{n/2}\delta^n/n\Gamma(n/2)$. Thus for significant intersection, we need to choose $t < 1/c$.

We performed the above calculations assuming that both $x + \delta B$ and $y + \delta B$ are both completely contained in the convex body K. Suppose instead that the intersection lies along the boundary of K as shown in Figure 4 For ease of notation, let $B_x = x + \delta B$ and $B_y = y + \delta B$. Since the local conductance is lower bounded by l, we have that for any $x \in K$, $\text{Vol}(B_x \cap K) \geq l\text{Vol}(B_x)$. Now, the volume of the intersection can be lower bounded as follows.

$$\begin{aligned}
\text{Vol}(B_x \cap B_y \cap K) &\geq \text{Vol}(B_x \cap K) - \text{Vol}(B_x \backslash B_y) \\
&= \text{Vol}(B_x \cap K) - \text{Vol}(B_x \backslash (B_x \cap B_y)) \\
&\geq l \cdot \text{Vol}(B_x) - ct \cdot \text{Vol}(\delta B) \\
&= (l - ct)\text{Vol}(\delta B)
\end{aligned}$$

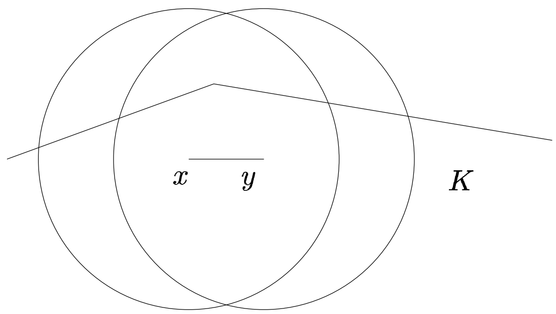

Figure 4: Volume of intersection when the intersection occurs along boundary.

We thus, have shown that $P_x(B_x \cap B_y \cap K) \geq l - ct$ and similarly $P_y(B_x \cap B_y \cap K) \geq l - ct$. Hence, for any set $S \subset K$ and points $x \in S, y \in \bar{S}$, if $|x - y| \leq t\delta/\sqrt{n}$, then $P_x(S) + P_y(\bar{S}) \geq l - ct$. This, implies the following lemma.

Lemma 2. *For any measurable set $S \subset K$, define*

$$S_1' = \left\{ x \in S | P_x(\bar{S}) < \frac{l - ct}{2} \right\}$$
$$S_2' = \left\{ x \in \bar{S} | P_x(S) < \frac{l - ct}{2} \right\}$$

Then, $d(S_1', S_2') > \frac{t\delta}{\sqrt{n}}$.

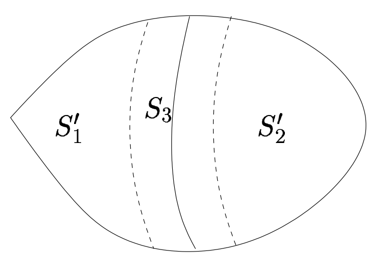

Figure 5: Partition of K into S_1', S_2' and S_3

We are now ready to prove the lower bound on the conductance Φ of the ball-walk.

Theorem 3. *For the ball-walk of step size δ in a convex body $K \subset \mathbb{R}^n$ of diameter D and local conductance bounded below by l, we have*

$$\Phi = \Omega\left(\frac{l^2\delta}{D\sqrt{n}}\right)$$

Proof. Let $S \subset K$ be the subset that attains the minimum conductance. For this S, define S_1' and S_2' as in Lemma 2. Also define $S_3 = K \backslash S_1' \backslash S_2'$. From the isoperimetry theorem proved earlier, we have that

$$\mathrm{Vol}(S_3) \geq \frac{d(S_1', S_2')}{D} \min\{\mathrm{Vol}(S_1'), \mathrm{Vol}(S_2')\}$$

Since, $Q(S) = \mathrm{Vol}(S)/\mathrm{Vol}(K)$, we have

$$Q(S_3) \geq \frac{d(S_1', S_2')}{D} \min\{Q(S_1'), Q(S_2')\}$$

From Lemma 2, we have $d(S_1', S_2') \geq t\delta/\sqrt{n}$. Hence,

$$\Phi(S) \geq \frac{t\delta(l - ct)}{2D \cdot 2\sqrt{n}} \min\{Q(S), Q(\bar{S})\}$$

Therefore,

$$\Phi \geq \frac{t\delta(l - ct)}{4D\sqrt{n}}$$

Thus, if we choose $t = l/2c$, we have the required result. □

Thus, from Theorem 3 and Theorem 1. we have that the mixing time of the ball walk is at most $O(D^2 n/\delta^2 l^4)$. In the next lecture, we will improve this result by proving that it suffices if average local conductance is bounded by l instead of worst case local conductance.

The bound of $O(D^2 n/\delta^2)$ is a lower bound on the number of steps for mixing in a ball-walk as demonstrated by the following example. Let the convex body K be a cylinder of height D. Consider the time taken to go from the top of the cylinder to the bottom. Each step in the ball-walk is roughly of length $\delta/2$. But only $1/\sqrt{n}$ fraction of the step is in the required direction. Hence, in each step of the ball-walk, we move a distance of $\delta/2\sqrt{n}$ in the required direction. We are thus in the case of random walk along a line where the step size is $\delta/2\sqrt{n}$. Hence, the mixing time is bounded below by $\Omega((D/(\delta/2\sqrt{n}))^2) = \Omega(D^2 n/\delta^2)$.