Homework 4

For this problem set, assume that circuits consist of $\vee$ and $\wedge$ gates with fanin 2 and the inputs are $x_1, x_2, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}$. In other words, the $\neg$ gates have been pushed down to the inputs. The depth of the circuit is the number of edges (wires) in the longest path from output to an input.

§1 Let $\mathcal{A}$ be a class of deterministic algorithms for a problem and I be the set of possible inputs. Both sets are finite. Let $\text{cost}(A, x)$ denote the cost of running algorithm A on input x. Prove Yao's lemma:

$$\max_{\mathcal{D}} \min_{A \in \mathcal{A}} \mathbf{E}_{x \in \mathcal{D}}[\text{cost}(A, x)] \quad = \quad \min_{\mathcal{P}} \max_{x \in I} \mathbf{E}_{A \in \mathcal{P}}[\text{cost}(A, x)],$$

where $\mathcal{D}$ is a probability distribution on I and $\mathcal{P}$ is a probability distribution on $\mathcal{A}$.

Does this result hold if the class of algorithms and inputs is infinite?

§2 For any graph G with n vertices, consider the following communication problem: Player 1 receives a clique C in G, and Player 2 receives an independent set I. They have to communicate in order to determine $|C \cap I|$. (Note that this number is either 0 or 1.) Prove an $O(\log^2 n)$ upperbound on the communication complexity.

Can you improve your upperbound or prove a lower bound better than $\Omega(\log n)$?

§3 Associate the following communication problem with any function $f : \{0,1\}^n \to \{0,1\}$. Player 1 gets any input x such that $f(x) = 0$ and player 2 gets any input y such that $f(y) = 1$. They have to communicate in order to determine a bit position i such that $x_i \neq y_i$.

Show that the communication complexity of this problem is *exactly* the minimum depth of any circuit that computes f.

§4 Use the previous question to show that computing the parity of n bits requires depth at least $2 \log n$.

§5 Show that in any circuit with m edges and depth d, there are $km/logd$ edges whose removal yields a circuit of depth less than $d/2^{k-1}$.