Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Classifying 2-manifolds** (two credits). Characterize the two surfaces depicted in Figure 1 in terms of genus, boundary, and orientability.

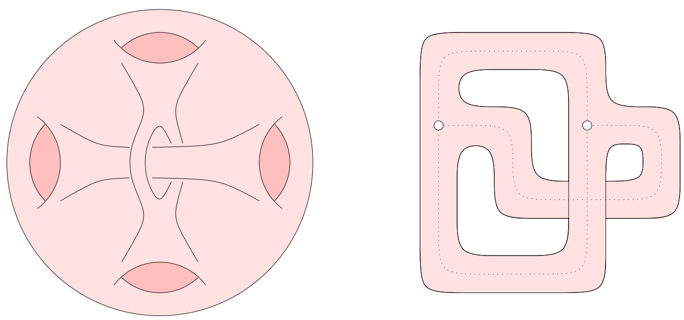

Figure 1: A 2-manifold without boundary on the left and one with boundary obtained by thickening a graph on the right.

2. **2-coloring** (two credits). Let K be a triangulation of an orientable 2-manifold without boundary. Construct L by decomposing each edge into two and each triangle into six. To do this, we add a new vertex in the interior of each edge. Similarly, we add a new vertex in the interior of each triangle, connecting it to the six vertices in the boundary of the triangle. The resulting structure is the same as the barycentric subdivision of K, which we will define in Chapter **??**.

   (i) Show that the vertices of L can be 3-colored such that no two neighboring vertices receive the same color.

   (ii) Prove that the triangles of L can be 2-colored such that no two triangles sharing an edge receive the same color.

3. **Klein bottle** (two credits). Cut and paste the standard polygonal schema for the Klein bottle (a, a, b, b) to obtain the polygonal schema in which opposite edges of a square are identified (a, b, a^{-1}, b); see Figure **??**.

4. **Triangulation of 2-manifold** (two credits). Let $V = \{1, 2, \ldots, n\}$ be a set of n vertices and $F \subseteq \binom{V}{3}$ a set of $\ell = \operatorname{card} F$ triangles. Give $\mathrm{O}(n+\ell)$-time algorithms for the following tasks:

(i) decide whether or not every edge is shared by exactly two triangles;

(ii) decide whether or not every vertex belongs to a set of triangles whose union is a disk.

5. **Intersection tests in** $\mathbb{R}^3$ (two credits). Let $a, b, c \in \mathbb{R}^3$ and $u, v, w \in \mathbb{R}^3$ be the vertices of two triangles in space. Write numerical tests for the following questions:

   (i) does u see a, b, c form a left-turn or a right-turn?

   (ii) does the line segment with endpoints u and v cross the plane that passes through a, b, c?

   (iii) are the boundaries of the two triangles linked in $\mathbb{R}^3$?

6. **Irreducible triangulations** (three credits). An *irreducible* triangulation is one in which every edge contraction changes its topological type. Prove that the only irreducible triangulation of $\mathbb{S}^2$ is the boundary of the tetrahedron, which consists of four triangles sharing six edges and four vertices.

7. **Graphs on Möbius strip** (one credit). Is every graph that can be embedded on the Möbius strip planar?

8. **Squared distance minimization** (two credits). Let S be a finite set of points in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x) = \sum_{p \in S} \|x - p\|^2$.

   (i) Show that f is a quadratic function and has a unique minimum.

   (ii) At which point does f attain its minimum?