III.3 Delaunay Complexes

In this section, we introduce a geometric constructions that limits the dimension of the simplices we get from a nerve. The main new structures are the Voronoi diagram and the Delaunay complex of a finite set of points. We begin by studying the inversion of space.

Inversion. Recall that $\mathbb{S}^d$ is the d-dimensional sphere with center at the origin and unit radius in $\mathbb{R}^{d+1}$. To invert $\mathbb{R}^{d+1}$, we map each point $x \neq 0$ to the point on the same half-line whose distance from the origin is one over the distance of x from 0. More formally, the *inversion* maps x to $\iota(x) = x/\|x\|^2$. It exchanges inside with outside and leaves points on $\mathbb{S}^d$ fixed. Clearly, $\iota(\iota(x)) = x$. We construct the image of a point x inside $\mathbb{S}^d$ by drawing right-angled triangles. First, we get a point $p \in \mathbb{S}^d$ such that $0xp$ has a right angle at x. Second, we choose x' on the half-line of x such that $0px'$ has a right angle at p. The angle at 0 is the same in both so the two triangles are similar. Hence, $\|x\| : \|p\| = \|p\| : \|x'\|$ which implies $\|x\|\|x'\| = \|p\|^2 = 1$ and thus $x' = \iota(x)$. We use this construction to show that the inversion maps spheres to spheres. We note, however, that it generally does not map centers to centers.

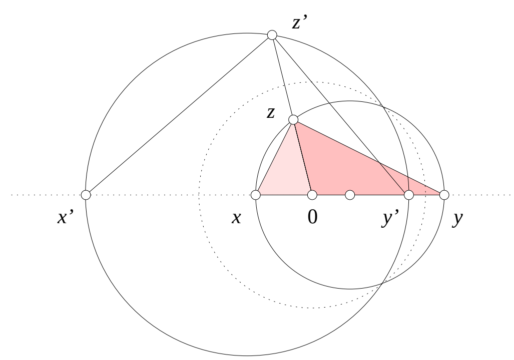

Figure III.1: As z sweeps out the circle passing through x and y, its image, $z' = \iota(z)$, sweeps out the circle passing through x' and y'.

Inversion Lemma. Let Σ be a d-sphere in $\mathbb{R}^{d+1}$. If $0 \notin \Sigma$ then $\iota(\Sigma)$ is a d-sphere and if $0 \in \Sigma$ then $\iota(\Sigma)$ is a d-plane.

Proof. Consider first the case in which Σ does not pass through the origin, as in Figure III.1. If 0 is the center of Σ then the result is obvious, so assume 0 is not the center. Draw the line passing through 0 and the center; it intersects Σ in points x and y, which we invert to get points $x' = \iota(x)$ and $y' = \iota(y)$. Let z be another point on Σ and $z' = \iota(z)$ its inverse. Then $\|x\|\|x'\| = \|z\|\|z'\| = 1$ which implies that the triangles $0xz$ and $0z'x'$ are similar. By the same token, $0yz$ and $0z'y'$ are similar. But xyz has a right angle at z implying the angles at x' and y' inside $x'y'z'$ add up to a right angle. It follows that $x'y'z'$ has a right angle at z'. As z travels on Σ, the sphere with diameter xy, the image z' travels on $\iota(\Sigma)$, the sphere with diameter $x'y'$. What happens when Σ passes through the origin, say $0 = x$? Then the triangle $0y'z'$ has a right angle at y'. Equivalently, the image of Σ is the plane normal to the vector y and passing through the point y'. ◻

The Inversion Lemma suggests we think of a d-plane as a special kind of d-sphere, namely one that passes through the point at infinity.

Stereographic projection. The inversion can be defined relative to any center $z \in \mathbb{R}^{d+1}$ and any radius $r > 0$, that is, $\iota_{z,r}(x) = r \cdot \iota(\frac{x-z}{r}) + z$. It is not difficult to check that x and $x' = \iota_{z,r}(x)$ indeed lie on the same half-line emanating from z and the product of their distances is $\|x - z\|\|x' - z\| = r^2$, as desired. We consider the special case in which the center is the point $N =$

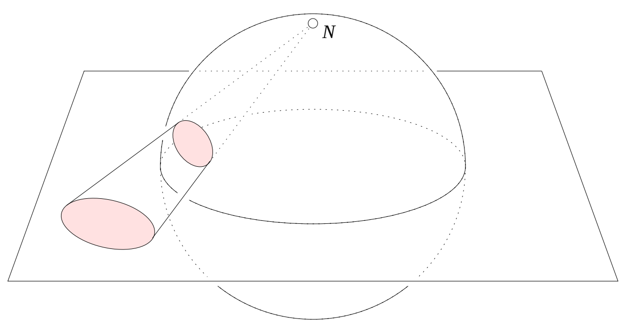

Figure III.2: The stereographic projection maps a circle on the unit sphere to a circle in the plane. If the circle on the sphere passes through the north-pole then its image is a line, that is, a circle that passes through the point at infinity.

$(0, \ldots, 0, 1)$, the north-pole of $\mathbb{S}^d$, and the radius is $r = \sqrt{2}$, the Euclidean distance between the north-pole and the equator. The image of $\mathbb{S}^d$ is the d-plane of points with vanishing $(d+1)$-st coordinates, which we denote as $\mathbb{R}^d$. The *stereographic projection* is the restriction of this particular inversion to the unit sphere, that is, $\varsigma : \mathbb{S}^d - \{N\} \to \mathbb{R}^d$ defined by $\varsigma(x) = \iota_{N,\sqrt{2}}(x)$, as sketched in Figure III.2. Similar to the inversion, the stereographic projection preserves spheres.

STEREOGRAPHIC PROJECTION LEMMA. Let Σ' be a $(d-1)$-sphere on $\mathbb{S}^d$. If $N \notin \Sigma'$ then $\varsigma(\Sigma')$ is a $(d-1)$-sphere and if $N \in \Sigma'$ then $\varsigma(\Sigma')$ is a $(d-1)$-plane in $\mathbb{R}^d$.

Indeed, every $(d-1)$-sphere considered in the lemma is the intersection of $\mathbb{S}^d$ with another d-sphere. Its image is therefore the intersection of $\mathbb{R}^d$ with the image of the d-sphere, which is either a d-sphere or a d-plane. The intersection is thus either a $(d-1)$-sphere or a $(d-1)$-plane. As before, we consider a plane as a special sphere that passes through the point at infinity.

Voronoi diagram. We use the stereographic projection and the more general inversion to elucidate the construction of a particular simplicial complex from a finite set $S \subseteq \mathbb{R}^d$. The *Voronoi cell* of a point u in S is the set of points for which u is the closest, $V_u = \{x \in \mathbb{R}^d \mid \|x-u\| \leq \|x-v\|,\ v \in S\}$. It is the

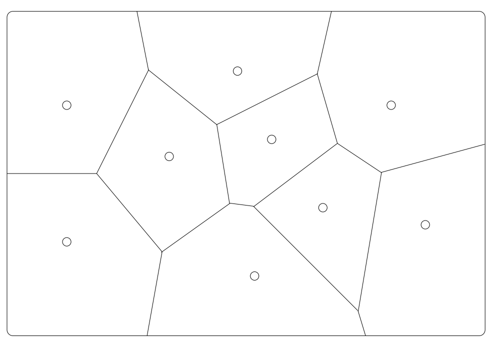

Figure III.3: The Voronoi diagram of nine points in the plane. By definition, each vertex of the diagram is equally far from the points that generate the incident Voronoi cells and further from all other points in S.

intersection of half-spaces of points at least as close to u as to v, over all points v in S. Hence, V_u is a convex polyhedron in $\mathbb{R}^d$. Any two Voronoi cells meet at most in a common piece of their boundary, and together the Voronoi cells cover the entire space, as illustrated in Figure III.3. The *Voronoi diagram* of S is the collection of Voronoi cells of its points.

We will shortly use a generalization of the concept to points u with real weights w_u. The *weighted squared distance*, or *power*, of a point $x \in \mathbb{R}^d$ from

u is then $\pi_u(x) = \|x - u\|^2 - w_u$. For positive weight, we can interpret the weighted point as the sphere with center u and square radius w_u. For a point x outside this sphere, the power is positive and equal to the square length of a tangent line segment from x to the sphere. For x on the sphere the power vanishes, and for x inside the sphere the power is negative. The *bisector* of two weighted points is the set of points with equal power from both. Just like in the unweighted case, the bisector is a plane normal to the line connecting the two points, except that is not necessarily halfway between them; see Figure III.4. Given a finite set of weighted points, we can thus define the *weighted Voronoi*

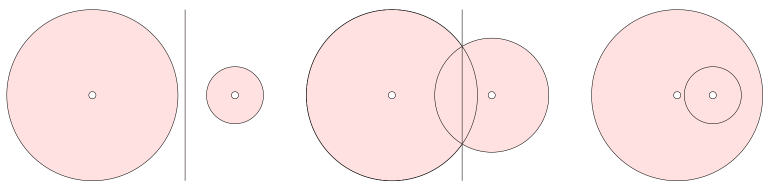

Figure III.4: The bisectors of pairs of weighted points. From left to right: two disjoint circles side by side, two intersecting circles, and two nested circles.

cell, or *power cell*, of u as the set of points $x \in \mathbb{R}^d$ with $\pi_u(x) \leq \pi_v(x)$ for all weighted points v in the set. Finally, the *weighted Voronoi diagram*, or *power diagram*, is the set of power cells of the weighted points.

Lifting. We get a different and perhaps more illuminating view of the Voronoi diagram by lifting its cells to one higher dimension. Let S be a finite set of points in $\mathbb{R}^d$, as before, but draw them in $\mathbb{R}^{d+1}$, adding zeros as $(d+1)$-st coordinates. Map each point $u \in S$ to $\mathbb{S}^d$ using the inverse of the stereographic projection, and let Π_u be the d-plane tangent to $\mathbb{S}^d$ touching the sphere in the point $\varsigma^{-1}(u)$, as illustrated in Figure III.5. Using inversion, we now map each d-plane Π_u to the d-sphere $\Sigma_u = \iota(\Pi_u)$. It passes through the north-pole and is tangent to $\mathbb{R}^d$, the preimage of $\mathbb{S}^d$. The arrangements of planes and of spheres are closely related to the Voronoi diagram. We focus on the spheres first.

First Sphere Lemma. A point $x \in \mathbb{R}^d$ belongs to the Voronoi cell of $u \in S$ iff the first intersection of the directed line segment from x to N is with the d-sphere Σ_u.

Proof. Interpret the sphere Σ_u as a weighted point, namely its center with weight equal to the square of its radius. The power of a point x is the squared length of a tangent line segment, which is equal to $\|x - u\|^2$ if $x \in \mathbb{R}^d$. It

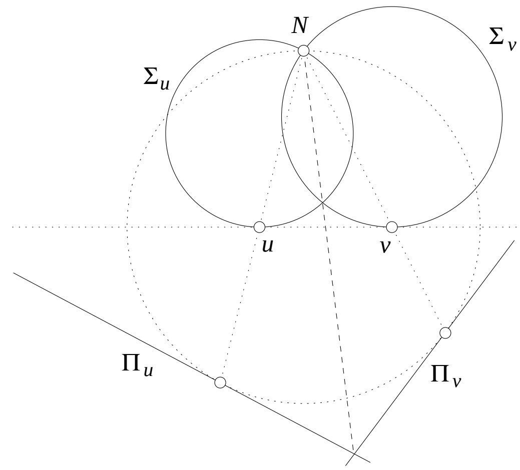

Figure III.5: We map the points u and v in $\mathbb{R}^1$ to the lines Π_u and Π_v tangent to $\mathbb{S}^1$ and further to the circles Σ_u and Σ_v passing through N and tangent to $\mathbb{R}^1$. The dashed line connecting N and the midpoint between u and v passes through the intersection of the two circles and the intersection of the two lines.

follows that the weighted Voronoi cell of the weighted center intersect $\mathbb{R}^d$ in the Voronoi cell of u. The claim follows because all bisectors of the weighted points pass through N. ⊒

Switching from spheres to planes we get a similar characterization of the Voronoi diagram in terms of tangent planes.

First Plane Lemma. A point $x \in \mathbb{R}^d$ belongs to the Voronoi cell of $u \in S$ iff the first intersection of the directed line segment from N to x is with the d-plane Π_u.

Delaunay triangulation. The *Delaunay complex* of a finite set $S \subseteq \mathbb{R}^d$ is isomorphic to the nerve of the Voronoi diagram,

$$\text{Delaunay} \quad = \quad \{\sigma \subseteq S \mid \bigcap_{u \in \sigma} V_u \neq \emptyset\}.$$

We say the set S is in general position if no $d+2$ of the points lie on a common $(d-1)$-sphere. This assumption implies that no $d+2$ Voronoi cells have a non-empty common intersection. Equivalently, the dimension of any simplex in the Delaunay complex is at most d. Assuming general position, we get a geometric realization by taking convex hulls of abstract simplices, as in Figure III.6. The result is often referred to as the *Delaunay triangulation* of S. To

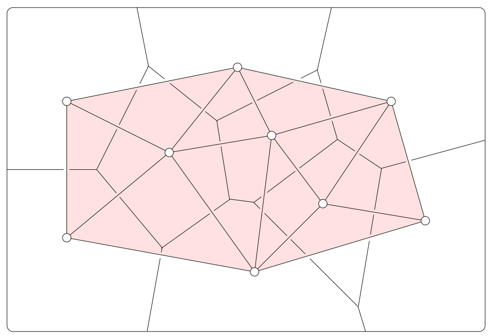

Figure III.6: The Delaunay triangulation superimposed on the Voronoi diagram. No four of the given points are cocircular implying the Delaunay complex has simplices of dimension at most two and a canonical geometric realization in $\mathbb{R}^2$.

see that this construction gives indeed a geometric realization of the Delaunay complex, we lift the points to the set $\varsigma^{-1}(S)$ on $\mathbb{S}^d$. Similarly, we lift a general point $x \in \mathbb{R}^d$ to the d-plane Π_x tangent to $\mathbb{S}^d$ at the point $\varsigma^{-1}(x)$. Keeping the same normal direction, we move this plane toward N. This corresponds to growing a $(d-1)$-sphere around x. The first point encountered by the plane corresponds to the first point encountered by the sphere, which is therefore the nearest to x. This suggests we add N to the set of lifted points and we take the convex hull in $\mathbb{R}^{d+1}$. The boundary of the resulting convex polytope consists of faces up to dimension d, some of which share N as a vertex. We are interested in the other faces, since they are spanned by points that correspond to Voronoi cells with a non-empty common intersection. Using central projection from N, we map these faces to $\mathbb{R}^d$. By convexity of the polytope, the images of the faces have no improper intersections. Indeed, we get the geometric realization of the Delaunay complex, as promised.

Similar to the Voronoi diagram, we can generalize the Delaunay complex to a finite set of points with real weights. Specifically, the *weighted Delaunay complex* is the abstract simplicial complex that contains a subset of the weighted points iff their weighted Voronoi cells have a non-empty common intersection. In contrast to the unweighted case, the cell of a weighted point can be empty, a difference that is sometimes overlooked. As a consequence, the vertex set of the weighted Delaunay triangulation is a subset and not necessarily the entire set of given weighted points. Assuming general position, this complex can again be geometrically realized by taking convex hulls of the abstract simplices. The appropriate notion of general position is that no point of $\mathbb{R}^d$ has the same power from more than $d + 1$ of the weighted points. This property is satisfied with probability one, a necessary requirement for a general position assumption.

Bibliographic notes. Voronoi diagrams are named after Georgy Voronoi [4] and Delaunay triangulations after Boris Delaunay (also Delone) [2]. Both structures have been studied centuries earlier by others, including Dirichlet, Gauß, and Descartes. Weighted Voronoi diagrams are perhaps as old as the unweighted ones and are known under a plethora of different names, including Thiessen polygons, Dirichlet tessellations, and power diagrams; see the survey article by Aurenhammer [1]. Their dual weighted Delaunay triangulations are also known under a variety of names, including regular triangulations and coherent triangulations; see e.g. [3].

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