3SUM-Hard Problems

Amit K. Patel

January 19, 2005

There are many problems for which the best known algorithms take time much greater than the worst known lower bound. This suggests the existence of more efficient algorithms. In computational geometry, one such barrier is the $\Theta(n^2)$ barrier. There exists problems for which the best algorithm runs in worst case time $\Theta(n^2)$. However, their lower bounds are much smaller. Designing faster algorithms for these problems can be frustrating so it helps to categorize them into a class of *difficult* problems.

I will describe a class of so called 3SUM-hard problems for which the problems are at least as hard as the following problem.

3SUM Given a set S of n integers, does there exist three elements $\{a, b, c\} \in S$ such that $a + b + c = 0$.

The best known algorithm for 3SUM runs in time $O(n^2)$ but the matching lower bound is known only for a restricted model of computation [Eri96]. It is not known whether $\Omega(n^2)$ is a lower bound for a more general model of computation (i.e. RAM).

A problem P is 3SUM-hard, if all instances I of 3SUM can be reduced to a constant number of P instances in time $f(n) \in o(n^2)$, where n is the size of I. P is then said to be at least as hard as 3SUM. One can not hope to find a subquadratic algorithm for P without finding a subquadratic algorithm for 3SUM.

For each of the following categories, I plan to show at least one example of a 3SUM-hard problem.

- **Incidence problems:** Given a set of objects, is there an object from a particular class that is incident with (or intersects) at least some given number of objects in the set.
- **Separator problems:** Given a set S of n objects in the plane, a line l is a separator if l does not intersect any object in S and the two halfplanes defined by l is non-empty. Problems in this category ask whether a separator exists for a given configuration of the plane.
- **Covering problems:** Does the union of a set of geometric objects fully cover a particular object.
- **Motion planning:** Given a starting configuration of a robot and a goal configuration, does there exist a path from start to goal avoiding collisions with a set of obstacles.

See [GO95] for a survey of 3SUM-hard problems.

References

[Eri96] Jeff Erickson. *Lower Bounds for Fundamental Geometric Problems.* PhD thesis, University of California at Berkeley, 1996.

[GO95] Anka Gajentaan and Mark H. Overmars. On a class of $O(n^2)$ problems in computational geometry. *Computational Geometry: Theory and Applications*, 5(3):165–185, 1995.