Matching via Power Diagrams

Jeff M. Phillips

A *Voronoi diagram* of a set of points (called sites) S is a partitioning of $\mathbb{R}^d$ into regions such that all points in a region are closer to a specific point s than any other point in S. Formally, for $s_i \in S$, region $R_i = \{q \in \mathbb{R}^d \mid d(q, s_i) < d(q, s_j) \forall s_j \in S \setminus s_i\}$. The *power diagram* [1] is a generalization of the Voronoi diagram where each site s_i has a weight w_i and the regions are now defined using a weighted distance: $P_i = \{q \in \mathbb{R}^d \mid d(q, s_i)^2 - w_i < d(q, s_j)^2 - w_j \forall s_j \in S \setminus s_i\}$.

Given two sets of points S and T where $|S| = |T| = n$, a bipartite matching m is a 1-1 assignment function $m : S \to T$. A natural question to ask is how fast can one solve

$$\min_m \sum_{i=1}^{n} d(s_i, m(s_i))^2$$

for the least squares bipartite matching. Aurenhammer, Hoffmann, and Aronov [2] demonstrate deep relationships between power diagrams and the least squares bipartite matching problem. They show a power diagram that represents the optimal bipartite matching that minimizes the sum of squared distances can be computed in $O(n^3 \log n)$ time.

More generally, when $|T| = m \geq n$ this result can be extended to where each site s_i has a capacity $c(s_i)$, where it accepts matches from $c(s_i)$ and so that $\sum_{i=0}^{n} c(s_i) = m$. This procedure takes $O(n^2 m \log m + nm \log^2 m)$ time. When the $T = \mathbb{R}^d$ and a probability distribution $\mu : S \to [0, 1]$ describes the capacity, then similarly there exists a power diagram to describe the assignment of the right measure of the space to each site and to minimize the squared distance, but this procedure is iterative and not combinatorial. Finding a combinatorial solution to question is an interesting open question.

I will give an introduction to power diagrams, bipartite matching, and explain the above result.

References

[1] Franz Aurenhammer. Voronoi diagrams—a survey of a fundamental geometric data structure.

[2] Franz Aurenhammer, Friedrich Hoffmann, and Boris Aronov. Mikowski-Type Theorems and Least-Squares Clustering. *Algorithmica*, 20:61–76, 1998.