Simple Analysis of Union-Find

Jeff M. Phillips

August 23, 2004

Abstract

Union-Find is a simple and incredibly useful algorithm for finding connected sets in a graph, $G = (V, E)$ that is often taught in an advanced algorithms class. Where $n = |V|$ and $m = |E|$, Union-Find has the famous, nearly-linear run-time bound of $O(m\alpha_T(m, n))$ where α_T is approximately the inverse of Ackermann's function. α_T is extremely slow growing–for any reasonable input it will never exceed 4. However, this result by Tarjan [4] is rarely presented because it is quite complicated. Instead an earlier known bound [2] of $O(m \log \log n)$ is often shown, as in CLR [1]. However, recently Seidel and Sharir [3], have given a much cleaner, easier to understand, and more elegant bound on Union-Find $O(m\alpha_S(m, n))$ with the same asymptotic complexity, but that does not rely on Ackermann's function.

In this talk I will give a brief introduction to Union-Find and its applications. Then I will present the new proof of Seidel and Sharir. The proof builds on itself recursively and I will prove bounds of $O(m \log n)$, $O(m \log \log n)$, and $O(m \log^* m)$ along the way. The result really is quite beautiful.

References

[1] Thomas H. Cormen, Charles E. Leiserson, and Ronald L. Rivest. *Introduction to Algorithms*. The MIT Press, 1985.

[2] John E. Hopcroft and Jeffrey D. Ullman. Set Merging Algorithms. *SIAM Journal of Computing*, 2(4):294–303, 1973.

[3] Raimund Seidel and Micha Sharir. Top-Down Analysis of Path Compression. *to appear SIAM Journal of Computing*, 2004.

[4] Robert E. Tarjan. Efficiency of a Good But Not Linear Set Union Algorithm. *JACM*, 22(2):215–225, 1975.