Linear and Hereditary Discrepancy

Jeff M. Phillips

Let P be a set of points. Let $\chi : P \to \{-1, +1\}$ be a coloring of P. Let $\mathcal{R} \subseteq 2^P$ be a set system defined on P. We can think of the points P as lying in $\mathbb{R}^d$ and we can think of $\mathcal{R}$ being all distinct ranges from a set of ranges (such as axis-aligned rectangles or unit circles) defined over P.

For finite dimension d, many sets of ranges have finite VC-dimension, and thus $|\mathcal{R}|$ is bounded by some polynomial of $|P|$.

We can define the discrepancy of P with respect to $\mathcal{R}$ as

$$\mathrm{disc}(\mathcal{R}) = \min_{\chi} \max_{R \in \mathcal{R}} |\chi(R)|$$

where we slightly abuse notation and let $\chi(R) = \sum_{p \in P \cap R} \chi(p)$. Intuitively, the discrepancy of a set system corresponds to the worst case coloring of P.

We can further define *linear* and *hereditary* discrepancy which are more general forms of discrepancy which give more information about the set systems. I will go over these definitions and discuss their associations and relevance. If things go well, I will also present the Beck-Fiala Theorem.