ApproxI: Introduction to LP-based approximation algorithms

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Overview

A combinatorial optimization problem is the problem of selecting an optimal solution from a finite set of feasible solutions. Many NP-hard combinatorial optimization problems can be attacked with **approximation algorithms** that yield in polynomial time a feasible solution whose objective function value is close enough to the optimal. Such approximation algorithms can be loosely categorized as combinatorial approximation algorithms and **LP-based approximation algorithms**. We are here interested in the latter category.

Many NP-hard combinatorial optimization problems can be rendered as an integer programming problem, which can be subsequently relaxed into a linear programming problem. However, the optimal solution to the linear programming problem in general does not coincide with the solution to the initial integer programming problem. Two basic techniques are widely exploited to derive a linear programming based approximation algorithm. The first technique is the **LP-based rounding**. It first solves the linear programming problem and then converts the fractional optimal solution to an integral solution. The second technique is known as **primal-dual schema**. This technique is based on the LP-duality theorem that a feasible solution to the dual programm provides a lower bound to the optimal solution to the primal linear program. The LP-duality theorem is also used in the **dual fitting** technique, a powerful tool for analyzing combinatorial approximation algorithms.

Set cover problem provides a particularly illustrative setting for introducing many vital concepts and techniques in approximation algorithms, and hence occupies a central place in the development of the field as well as in this talk. Given a universe of a finite number of elements $U = \{e_1, e_2, \ldots, e_n\}$, a collection of subsets of U, $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$, and a cost function $c : \mathcal{S} \to Q^+$, the set cover problem aims to find a minimum cost subcollection of $\mathcal{S}$ that covers all the elements in U.

A combinatorial approximation algorithm [7, 8, 3], a rounding based approximation algorithm [5], and a primal-dual approximation algorithm [2] to the set cover problem will be described. The combinatorial approximation algorithm is also analyzed with the dual fitting technique [8, 3].

Notes on the talk and references

Since this talk is intended to be the first one in a series of talks on approximation algorithms (by me or other willing speakers), the emphasis is on constructing a big picture view of the field by conveying intuition behind fundamental concepts and techniques. As such, we will focus on simple and illustrative examples in this talk, and defer to subsequent talks the application of these powerful techniques in a more elaborate way.

Rounding, especially random rounding, was covered by Nabil Mustafa in a previous talk [9], and hence will only receive brief treatment in this talk.

I assume an audience with a decent background of algorithm analysis and computation complexity. However, background in approximation algorithms and LP duality is not assumed. To benefit an audience with diverse background in this field, I might spend some time in explaining concepts and techniques that you may already be familiar with. If so, feel free to ignore me and work on your own stuff. On the other hand, if I use some terms without appropriate explanation and confuse you, be sure to stop me and ask.

The primary reference book for this talk and some subsequent talks on this topic is Vazirani [10]. Chapters covered in this talk include chapters 1, 2, 12, 13, 14, and 15. For more books on approximation algorithms, see Hochbaum [6] and Ausiello [1]. For an introduction to linear programming, see Chvatal [4].

References

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[10] V. Vazirani. *Approximation Algorithms*. Springer-Verlag Berlin Heidelberg New York, second edition, 2003.