Arrangements: Combinatorial and Algorithmic Applications

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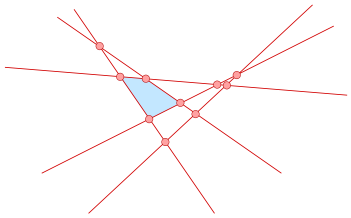
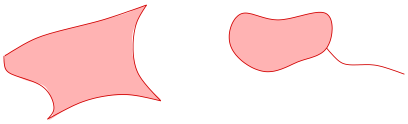
Arrangements



- ☆ $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$: Geometric objects in $\mathbb{R}^d$
  hyperplanes, hypersurfaces, surface patches.
- ☆ *Arrangement* of Γ, $\mathcal{A}(\Gamma)$:
  - Subdivision of $\mathbb{R}^d$ into maximal connected regions, called *cells*.
  - Each region lies in the same subset of Γ.
- ☆ *k-cell*: k-dimensional cell of $\mathcal{A}(\Gamma)$.
- ☆ *Combinatorial complexity* of $\mathcal{A}(\Gamma)$:
  # cells of all dimensions in $\mathcal{A}(\Gamma)$.

Assumptions on Γ

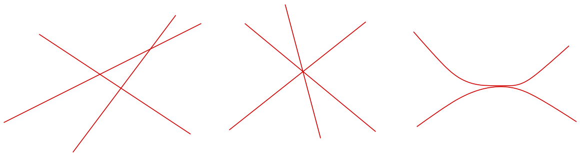
- γ_i is a *semialgebraic* set
  - Boolean formula on polynomial equalities and inequalities; $\{f = 0\}, \{f > 0\}$.
- *constant description complexity*.
  - Constant size formula.
  - Degree of each polynomial $\leq b$ (constant).
  - *Segments, circular arcs, conic sections.*
- *Local dimension* of every point is $d - 1$.



- d constant; n is large. Measure complexity as a function of n.
- Γ in *general position*: slight perturbation does not alter the combinatorial structure of $\mathcal{A}(\Gamma)$.

General Position Assumption

- Rules out degenerate configurations.
  - Intersection of $d + 1$ surfaces is empty,
  - common intersection of k surfaces is a $(d - k)$-dimensional region,
  - all intersections are transversals, ...



- Degeneracies are either handled directly or using *perturbation schemes*.
- *Numerical perturbations*
- *Symbolic perturbations*
  - Use infinitsmals.
  - *How many infitismals are required?*

Arrangements of Hyperplanes: Early Results

- ☆ J. Steiner, 1826: Complexity of arrangements of lines, planes, circles, and spheres under general position.
- ☆ S. Roberts, 1888: Generalized the argument to degenerate arrangements.
- ☆ R. Buck, 1943: Complexity of hyperplane arrangements.

$$\#k\text{-cells} \leq \binom{n}{d-k} \sum_{i=0}^{k} \binom{n-d+k}{i}.$$

- ☆ T. Zaslavsky, 1975: Complexity of degenerate arrangements.

On Arrangements

"Today I would like to survey the somewhat related field of arrangements of hyperplanes, which I expect to become increasingly popular during the next few years. … the theory of arrangements may be developed, much like, topology, in rectilinear or curved versions as well as in discrete or continuous variants, and that in these developments it impinges upon many aspects of convexity, topology, and geometry which seemed to be quite unrelated."

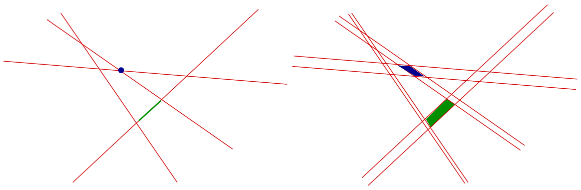
Branko Grünbaum, 1971

On Arrangements

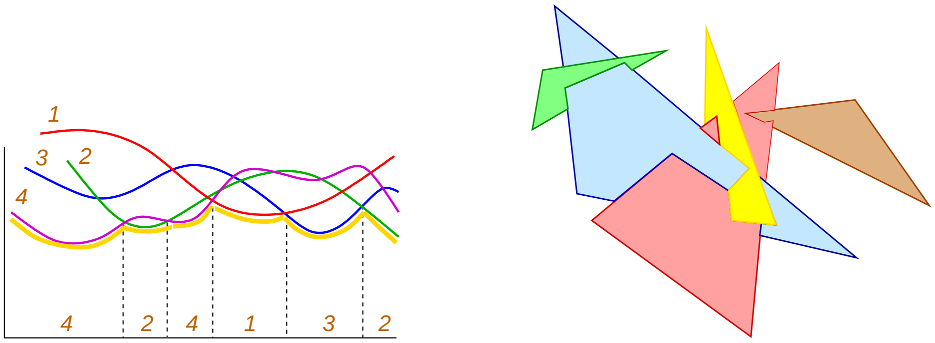
> *. . . led me to believe that arrangements of hyperplanes are at the very heart of computational geometry — and this is my belief now more than ever.* *Herbert Edelsbrunner 1987*

Arrangements of Surfaces

- ☆ # d-cells: $O\left(\left(\frac{nb}{d}\right)^d\right)$ [Warren, 1968]
- ☆ # cells of all dimensions $O\left(\left(\frac{nb}{d}\right)^d\right)$ [Pollack-Roy, 1993]
  - Replace each hypersurface by two parallel copies of the surface very close to each other.
  - Each k-face becomes a d-face.



Parts of Arrangements

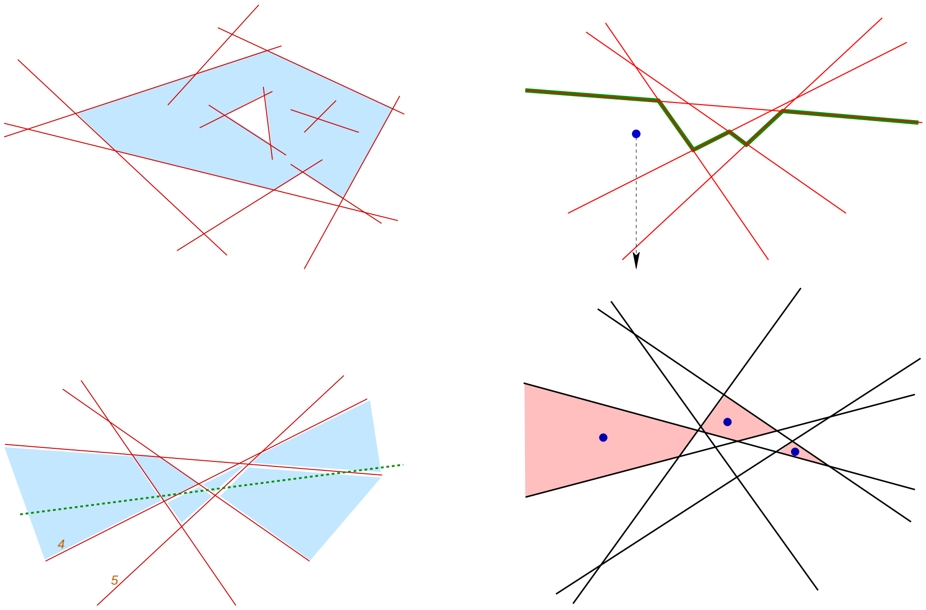


Lower Envelope

Several Applications:

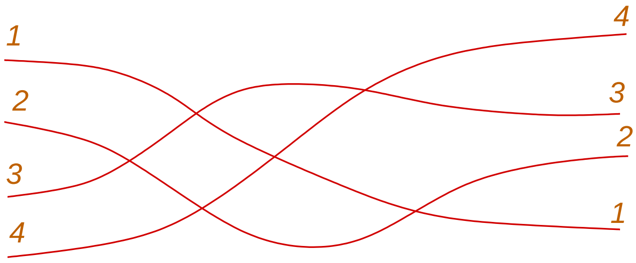
- ☆ Geometric Transversals
- ☆ Voronoi diagrams
- ☆ Motion planning

Parts of Arrangements



[A.-Sharir, 2000]

Arrangements of Pseudo-Lines



- ☆ Family of pseudolines: Collection of curves in which each pair meets *at most* once, and crosses there
- ☆ Allowable sequences: Introduced by [Goodman-Pollack, 1980]
  Allowable sequences can be realized by pseudoline arrangements
  $1234 \xrightarrow{23} 1324 \xrightarrow{13} 3124 \xrightarrow{24} 3142 \xrightarrow{14} 3412 \xrightarrow{12} 3421 \xrightarrow{34} 4321$
- ☆ Arrangement *strechable*: Isomporphic to one produced by an arrangement of lines

Pseudoline Arrangements: Combinatorial Issues

- ☆ Lower envelope $O(n)$
- ☆ Single cell $O(n)$
- ☆ Zone of a pseudoline $O(n)$
- ☆ k-Level $O(nk^{1/3})$
- ☆ $\leq k$-Level $O(nk)$
- ☆ Incidences with m points: $O(m^{2/3}n^{2/3} + n)$

- ☆ m distinct faces $O(m^{2/3}n^{2/3} + n)$

  ⋮

Pseudoline Arrangements: Algorithmic Issues

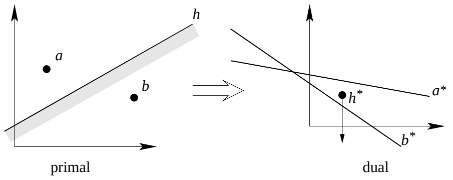
- ☆ Model of computation:
  - Curves are graphs of fixed-degree polynomials
  - Roots of such polynomial can be computed in $O(1)$ time
- ☆ Lower envelope, single cell, zone:
  Algorithms for lines extend
- ☆ Hopcroft's Problem: P: Set of m points, Γ: Set of n pseudolines
  Report all incidences between p and Γ!

  Γ: Set of lines ⇒ Time: $m^{2/3}n^{2/3} + m + n$

Incidences: Algorithm for Lines

Works in two stages:

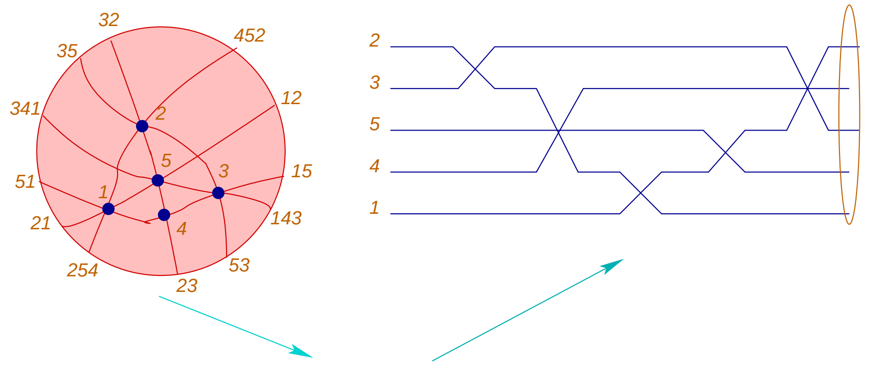
- ☆ First develop an $m^2 + n$ algorithm
  - duality transform + point-location in dual arrangement



  - Compute the dual arrangement and do point location
- ☆ Create $O(r^2)$ subproblems, each involving $\leq n/r$ lines and m/r^2 points
  - Compute $(1/r)$-cutting $\Delta = \{\Delta_1, \dots, \Delta_u\}$, $u = O(r^2)$
  - L_i: Lines intersecting Δ_i, $P_i = P \cap \Delta_i$
  - Compute incidences between L_i, P_i using the above algorithm
    Total time: $m^2/r^2 + nr$
  - Choose $r = m^{2/3}/n^{1/3}$, time: $m^{2/3}n^{2/3} + m + n$

Duality for Pseudolines

[Goodman, 1980]


32
452
35
12
341
2
5
3
15
51
1
21
143
4
254
53
23
2
3
5
4
1

$23541 \xrightarrow{23} 32541 \xrightarrow{254} 34521 \xrightarrow{21} 34512 \xrightarrow{51} 34152 \xrightarrow{341} 14352 \xrightarrow{35} 14532$

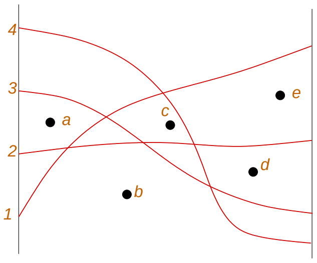
Computing the dual arrangment:

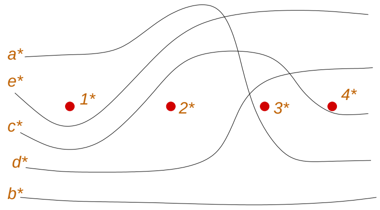
Simple algorithm: mn time

How to compute the dual arrangement in $m^2 + n$ time?

Duality for Pseudolines

[A.-Sharir, 2001]

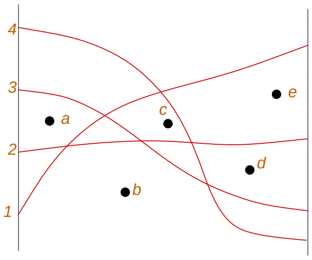


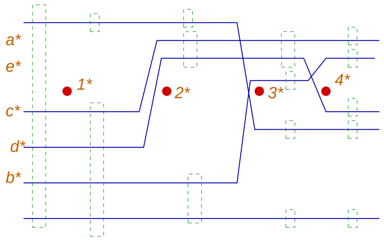


- ☆ Draw L^* on the x-axis sorted by their intercepts
- ☆ $p, q \in P$: $p \prec q$ if the "lowest" pseudoline that separates p and q lies below q and above p

  Lemma: *$\prec$ is a total order on P*
- ☆ Use $\prec$ to sort P^* at $x = -\infty$ and draw them as a wiring diagram

Computing the Dual Arrangement



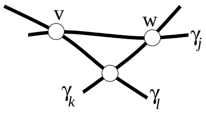


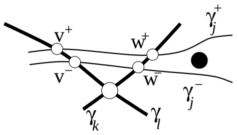
- ☆ Sweep-line algorithm
- ☆ Maintain P as a set of bundles; two points in the same bundle if they have not been separated by any line of L visited so far
- ☆ Bundles are ordered in the y-direction
- ☆ Maintain the arrangement of *bundles*
- ☆ At the end each bundle has only point

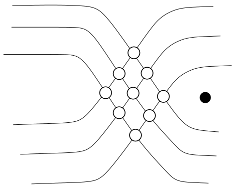
Computing the Dual Arrangement

Processing ℓ_i^*

- ☆ Split each bundle B into two subsets B^+, B^-
  $B^+ \subseteq B$: Points lying *above* ℓ_i
  $B^- \subseteq B$: Points lying *below* ℓ_i
- ☆ $\{B_1, \ldots, B_k\} \longrightarrow \{B_1^-, \ldots, B_k^-, B_1^+, \ldots, B_k^+\}$
  Keep only nonempty subsets
- ☆ Update the arrangement of bundles

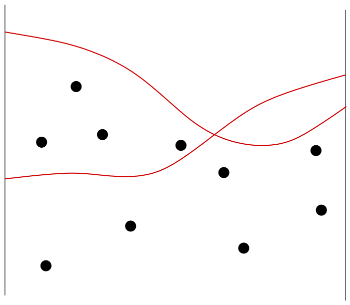

v
w
γ_j
γ_k
γ_l


v^+
w^+
γ_j^+
v^-
w^-
γ_j^-
γ_k
γ_l



Computing Dual Arrangement: Data Structure

- ☆ EMPTY (γ)
- ☆ INSERT (p)
- ☆ DELETE (p)
- ☆ SPLIT (γ)



Cost: m^{ε} time

Total time in computing $\mathcal{A}(P^*)$: $O((m^2+n)m^{\varepsilon})$

Computing incidences between P and L: $m^{2/3}n^{2/3}+m+n$

Computing m marked faces: $m^{2/3}n^{2/3}+m+n$

Computing m marked faces in n unit circles: $m^{2/3}n^{2/3}+m+n$

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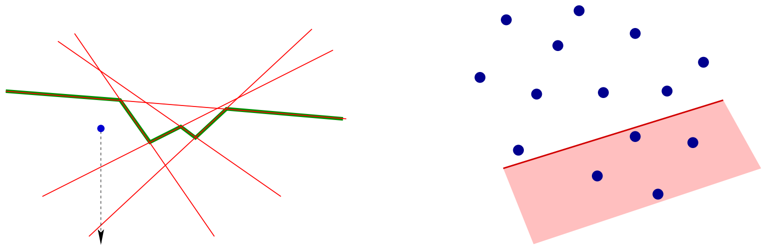
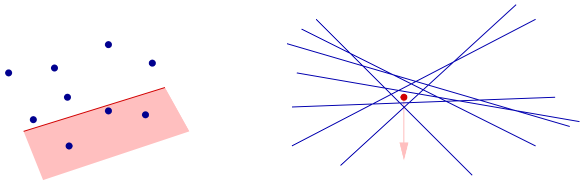
k-Level and k-Set



- ☆ k-level: Set of edges that lie above k lines
  $\leq k$-level: Portion of arrangement at level $\leq k$
- ☆ k-set: A k-subset of P that can be separated by a line
  Best-known upper bound: $nk^{1/3}$ [Dey, 1998]
- ☆ $\leq k$-set: $\leq k$ size subset of P that can be separated by a line
  First upper bound: $2nk-k^2-k$ [Goodman-Pollack, 1984]
  Tight bound: $nk+1$ [Alon-Györi, 1986]
  Higher dimensions: $n^{\lfloor d/2\rfloor}k^{\lceil d/2\rceil}$ [Clarkson-Shor, 1989]
- ☆ Complexity of a typical level: n

Halfplane Range Reporting



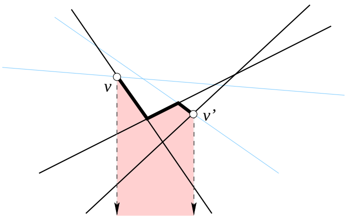
☆ Preprocess P into a data structure so that all points lying in query halfplane can be reported quickly

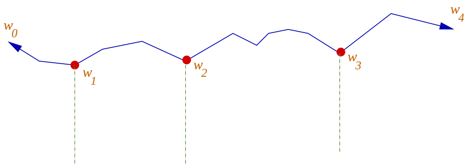
☆ Dual: Report all lines that intersect a vertical ray

☆ [Chazelle, Guibas, Lee, 1985]
Space: $O(n)$; Query: $O(\log n + k)$

☆ [A., Arge, Erickson, Franciosca, Vitter, 1998]
- Simpler data structure with the same bound
- Extends to the I/O model: disk block size b
  Space: $O(n/b)$ disk blocks; Query: $O(\log_b n + k/b)$ I/Os

Halfplane Range Reporting





☆ λ_k: k-level of $\mathcal{A}(L)$

☆ *Cluster* induced by v, v': Set of lines that appear below λ_k between v, v'

☆ *Clustering* λ_k: $\langle w_0, w_1, \ldots, w_u\rangle$ a subsequence of vertices of λ_k
- $\mathcal{C} = \langle C_1, \ldots, C_u\rangle$: C_i cluster induced by w_{i-1} and w_i
- b-clustering: $\max |C_i| \le b$

☆ **Claim:** *Greedy $3k$-clustering of λ_k has $O(n/k)$ clusters.*

Halfplane Range Reporting

Preprocessing (L)

```
$i = 1$, $r = \log n$
while $L \neq \emptyset$
    Compute $\lambda_r$: $r$-level of $\mathcal{A}(L)$
    Compute $(3r)$-clustering $\mathcal{C}_i$ of $\lambda_r$
    $L_i$: Lines that appear below $\lambda_r$
    $L = L \setminus L_i$; $i = i + 1$
end while
```

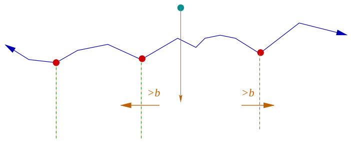
Query: p be the query point

```
repeat
$i = i + 1$; $O_i =$ Report $(\mathcal{C}_i, p)$
until $|O_i| < b$
```

Size

RAM: n

I/O model: n/b blocks



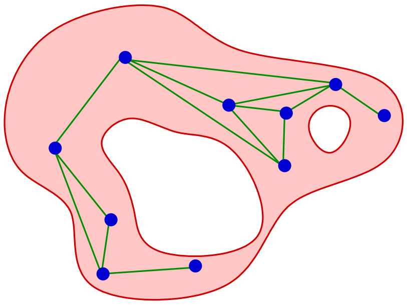
Query time: $\log n + k$

$\log_b n + k/b$ I/Os

Furtther Issues

☆ Arrangements in very high dimensions

- Too expensive to compute exactly
- Approximate computation



☆ Voronoi diagram of lines in $\mathbb{R}^3$