Due Date: April 23, 2001

Problem 1: Let A and B be two languages. Show that if $A \leq_m^{\mathsf{LOG}} B$ and $B \in \mathsf{LOG}$, then $A \in \mathsf{LOG}$. Rceall that $\mathsf{LOG} = \mathsf{DSPACE}(\log n)$.

Problem 2: Let M be an alternating Turing machine such that every computation path on any string of length n uses at most $S(n)$ space and at most $A(n)$ alternations. Show that $L(M)$ can be decided by a deterministic TM using $O(A(n)S(n) + S^2(n))$ space.

Problem 3: Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function. Show that there exists a circuit of $O(2^n/n)$ size and $O(\log n)$ depth that computes f.
(**Hint:** *Express f as $f(x_1, \ldots, x_n) = \bigvee_i y_{i_1} y_{i_1} \cdots y_{i_k} f_i(x_{k+1}, \ldots, x_n)$, choose an appropriate value of k, and show that the circuit for this expression requires the desired size.*)

Problem 4: The Boolean hierarchy is defined as follows.

$$\begin{aligned} \Xi_0\mathsf{P} &= \mathsf{P}, \\ \Xi_{2i+1}\mathsf{P} &= \{L_1 \cup L_2 \mid L_1 \in \Xi_{2i}\mathsf{P},\ L_2 \in \mathsf{NP}\}, \\ \Xi_{2i+2}\mathsf{P} &= \{L_1 \cap L_2 \mid L_1 \in \Xi_{2i+1}\mathsf{P},\ L_2 \in \mathsf{co-NP}.\} \end{aligned}$$

A language L is in the class DP if there exist two languages $L_1, L_2 \in \mathsf{NP}$ such that $L = L_1 \cap \bar{L}_2$.

Notice that $\Xi_1\mathsf{P} = \mathsf{NP}$ and $\Xi_2\mathsf{P} = \mathsf{DP}$. Set $\mathsf{BH} = \bigcup_{i \geq 0} \Xi_i\mathsf{P}$.

Prove that any language $L \in \mathsf{BH}$ can be represented as

$L = L_1 \cup L_2 \cup \cdots L_u$, where each $L_i \in \mathsf{DP}$. (That is, every language in BH is a finite union of DP sets.)

Problem 5: Let NC^i (resp. AC^i) denote the set of languages accepted by a $O(\log^i n)$-depth, $n^{O(1)}$-size ciruit, each of whose gate has bounded (resp. unbounded) fain-in.

(i) For any $i \geq 0$, Show that $\mathsf{AC}^i \subseteq \mathsf{NC}^{i+1} \subseteq \mathsf{AC}^{i+1}$.

(ii) Prove that if $\mathsf{NC}_{i+1} = \mathsf{NC}_i$, then $\mathsf{NC} = \mathsf{NC}_i$.