Decision Theory

CSCI 2951-F

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Decision Theory

What does it mean to make an optimal decision?

- Asked by economists to study consumer behavior
- Asked by MBAs to maximize profit
- Asked by leaders to allocate resources
- Asked in OR to maximize efficiency of operations
- Asked in AI to model intelligence

- Asked (sort of) by any intelligent person every day

Utility Functions

- A *utility function* is a mapping from world states to real numbers
- Sometimes called a *value function*
- Rational or optimal behavior is typically viewed as maximizing expected utility:

$$\max_a \sum_s P(s \mid a) U(s)$$

a = actions, s = states

Are Utility Functions Natural?

- Some have argued that people don't really have utility functions
  - What is the utility of the current state?
  - What was your utility at 8:00pm last night?
  - *Utility elicitation* is difficult problem

- It's easy to communicate *preferences*
- Theorem (sorta): Given a plausible set of assumptions about your preferences, there must exist a consistent utility function

Axioms of Utility Theory

Bet/gamble between A and C

- Orderability: $(A \succ B) \vee (A \prec B) \vee (A \sim B)$
- Transitivity: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
- Continuity: $A \succ B \succ C \Rightarrow \exists p [p, A; 1-p, C] \sim B$
- Substitutability: $A \sim B \Rightarrow [p, A; 1-p, C] \sim [p, B; 1-p, C]$
- Monotonicity: $A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; 1-p, B] \succeq [q, A; 1-q, B])$
- Decomposability:
  $[p, A; (1-p), [q, B; (1-q), C]] \sim [p, A; (1-p)q, B; (1-p)(1-q), C]$

Consequences of Preference Axioms

- Utility Principle
  - There exists a real-valued function U:

$U(A) > U(B) \Leftrightarrow A \succ B$ ← A preferred to B

$U(A) = U(B) \Leftrightarrow A \sim B$ ← Indifferent between A and B

- Expected Utility Principle
  - The utility of a lottery can be calculated as:

Lottery that results in S_i with prob p_i → $U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)$

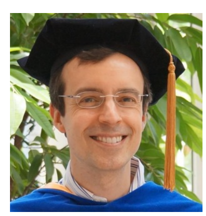
More Consequences

- Scale invariance

- Shift invariance

Maximizing Utility

- Suppose you want to be famous
- You can be either (N,M,C)
  - Nobody
  - Modestly Famous
  - Celebrity
- Your utility function:
  - U(N) = 20
  - U(M) = 50
  - U(C) = 100
- You have to decide between going to grad school and becoming a professor (G) or going to Hollywood and becoming an actor (A)



Outcome Probabilities

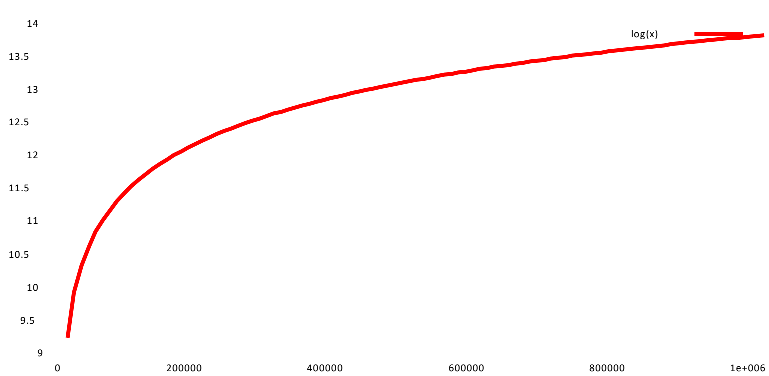
- P(N|G)=0.5, P(M|G)=0.4, P(C|G)=0.1
- P(N|H)=0.6, P(M|H)=0.2, P(C|H)=0.2
- Maximize expected utility:
  - U(N) = 20, U(M) = 50, U(C) = 100

$$EU_G = 0.5(20) + 0.4(50) + 0.1(100) = 40$$
$$EU_H = 0.6(20) + 0.2(50) + 0.2(100) = 42$$

Hollywood wins!

Utility of Money

- How much happier are you with an extra $1M?
- How much happier is Jeff Bezos with an extra $1M?
- Some have proposed:

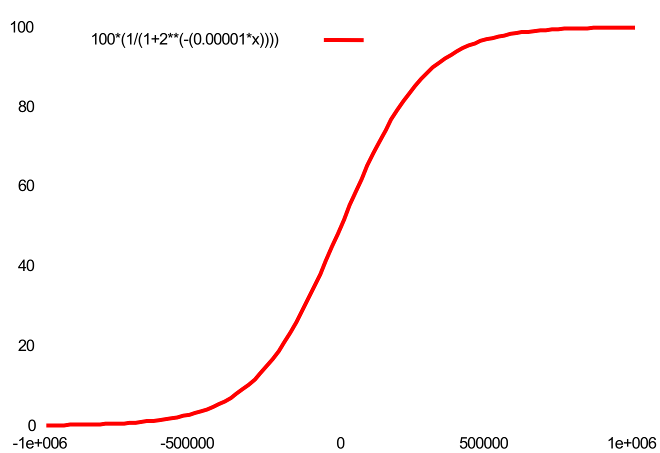


Utility of Money

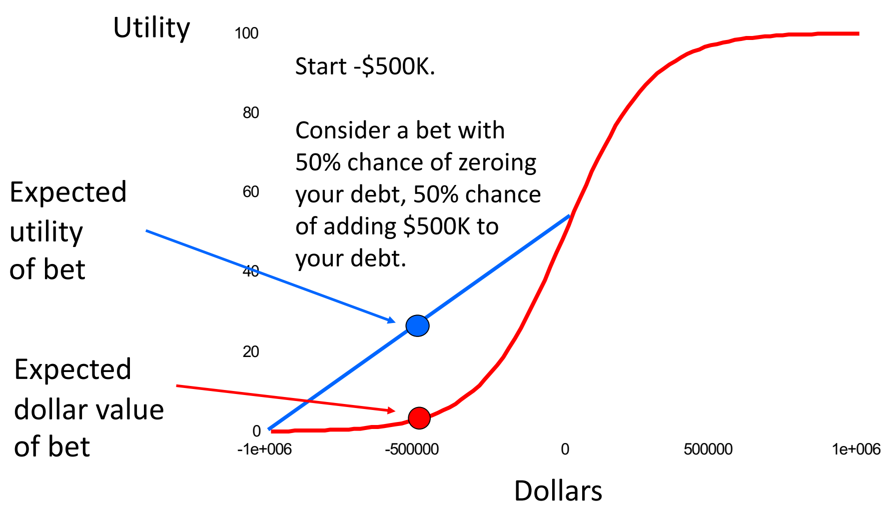
- U(money) should drop slowly in negative region too
- If you're solvent, losing $1M is pretty bad
- If already $10M in debt, losing another $1M isn't that bad
- Utility of money is probably sigmoidal (S shaped)

A Sigmoidal Utility Function

$$U(\$X) = 100\frac{1}{1+2^{-0.00001X}}$$



Sigmoidal Utility and Gambling



Bonus Material (if time permits)

Value of Information

- Many choices are choices about acquiring information:
  - Medical tests (x-rays, CT-scans, mammograms, etc.)
  - Pregnancy tests
  - Pre-purchase house/car inspections
  - Hiring consultants
  - Hiring a trainer
  - Funding research
  - Checking one's own credit score
  - Checking somebody else's credit score
  - Background checks
  - Drug tests
  - Real time stock prices
- Information is rarely free.
- How do we determine what it is worth to us?

Value of Information

- Expected utility of action a with evidence E:

$$\mathrm{EU}_{\mathrm{E}}(A \mid E) = \max_{a \in A} \sum_i P(S_i \mid E, a) U(S_i)$$

- Expected utility given new evidence E' - **after** E' is known

$$\mathrm{EU}_{E,E'}(A \mid E, E') = \max_{a \in A} \sum_i P(S_i \mid E, E', a) U(S_i)$$

- Expected value of knowing E' (**V**alue of **P**erfect **I**nformation) – **before** E' is known

$$\mathrm{VPI}_E(E') = \left(\sum_{E'} P(E' \mid E) \mathrm{EU}_{E,E'}(A' \mid E, E') \right) - \mathrm{EU}_{\mathrm{E}}(A \mid E)$$

Expected utility given New information (weighted)

Previous Expected utility

VPI Example

- Should you pay to subscribe for traffic information? Assume:
  - Time = cost = -utility
  - Cost of taking highway to work (w/o traffic_jam) = 15
  - Cost of taking highway to work (w/traffic_jam) = 30
  - Cost of taking local roads to work = 20
  - P(traffic_jam) = 0.15

- Steps:
  - Determine optimal decision w/o information: EU(A|{})
  - Determine optimal decisions given information: $EU_T(A|T)$
  - Compute expected value of optimal decisions given T
  - Estimate value of information (difference in prev. slide)

VPI for Traffic Info

- Cost of local roads = 20
- Cost of highway = 0.15*30 + 0.85*15 = 17.25

} EU(A|{})

- Traffic = true case: Take local roads; cost = 20
- Traffic = false case: Take highway; cost = 15

} $EU_T(A|T)$

- Expected cost: 0.15*20 + 0.85*15 = 15.75 } Expectation
- Value = 1.5 } VPI

- **Important:** In this case, the optimal choice given the information was trivial. In general, we may to do more computation to determine the optimal choice given new information – not all decisions are "one shot"

Properties of VPI

- VPI is non-negative!
- VPI is not additive
- VPI is easy to compute and is often used to determine how much you should pay for ***one*** extra piece of information. Why is this myopic?

For example, knowing X AND Y together may useful, while knowing just one alone may be useless.

More Properties of VPI

- Acquiring information optimally is very difficult
- Need to construct a conditional plan for every possible outcome before you ask for even the first piece of information
  - Suppose you're a doctor planning to treat a patient
  - Picking the optimal test to do first requires that you consider all subsequent tests and all possible treatments as a result of these tests
- General versions of this problem are intractable!