Topic 11: Dynamic Programming

(CLRS 15.0–15.2)

CPS 230, Fall 2001

1 Dynamic programming

- We have previously discussed how divide-and-conquer can often be used to obtain efficient algorithms.
  - Examples: matrix multiplication, merge sort, quicksort,....
- Sometimes direct use of divide-and-conquer does not yield efficient algorithms—in fact, sometimes it results in really bad algorithms.
- Today we will discuss a technique which can often be used to improve upon an inefficient divide-and-conquer algorithm.
  - The technique is called "Dynamic programming" for historical reasons. It really is neither especially "dynamic" nor especially "programming" related.
  - We will discuss dynamic programming by looking at an example.

1.1 Computing Fibonacci numbers

- The Fibonacci numbers F_n are defined by the well-known recurrence

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases} \tag{1}$$

  The sequence for $n = 0, 1, 2, 3, \ldots$ is $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.

- The sequence is related to rabbits, how needles arrange themselves on pine cones, aesthetics, and graduate students, among other things.
- A naive recursive program to compute Fibonacci numbers based upon (1) would require $O(F_n)$ time, which is exponential in n!
- The reason is that the same subproblem is solved multiple times. For example, to compute $F(6)$, we need to compute $F(5)$ and $F(4)$. To compute $F(7)$, we recompute $F(6)$ and $F(5)$. The waste compounds recursively!

- The obvious remedy is to store the value of each Fibonacci number in a table when it is computed. Next time it is needed, we won't have to recompute it from scratch, but rather we can just look up its value in the table.
$\Longrightarrow$ $O(n)$ time to compute F_n!!!

- As an aside, using generating functions or the trial-and-error method (see earlier handout), we can derive a closed-form expression for F_n:

$$F_n = \frac{1}{\sqrt{5}}(\Phi^n - \widehat{\Phi}^n), \tag{2}$$

where $\Phi = (1+\sqrt{5})/2 \approx 1.61803\ldots$ and $\widehat{\Phi} = -1/\Phi \approx -0.61803$ are the solutions to the equation $1 - z - z^2 = 0$. We refer to Φ as the golden ratio. Note that the second term in (2) is exponentially small. We can compute F_n simply by rounding $\Phi^n/\sqrt{5}$ to the nearest integer.

- Therefore, using a table to store subproblem solutions reduces the running time from $O(\Phi^n)$ to $O(n)$. Pretty big improvement!

1.2 Matrix-chain multiplication

- Problem: Given a sequence of matrices $A_1, A_2, A_3, ..., A_n$, find the best way (using the minimal number of multiplications) to compute their product.
  - Isn't there only one way? $((\cdots((A_1 \times A_2) \times A_3) \times \cdots) \times A_n)$
  - No, matrix multiplication is *associative*;
    e.g., $A_1 \times (A_2 \times (A_3 \times (\cdots \times (A_{n-1} \times A_n)\cdots)))$ yields the same matrix.
  - Different multiplication orders do not cost the same:
    * Multiplying $p \times q$ matrix A and $q \times r$ matrix B takes $p \cdot q \cdot r$ multiplications; the result is a $p \times r$ matrix.
    * Consider multiplying 10×100 matrix A_1 with 100×5 matrix A_2 and 5×50 matrix A_3.
      - $(A_1 \times A_2) \times A_3$ takes $10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 7500$ multiplications.
      - $A_1 \times (A_2 \times A_3)$ takes $100 \cdot 5 \cdot 50 + 10 \cdot 50 \cdot 100 = 75000$ multiplications.

- In general, let A_i be a $p_{i-1} \times p_i$ matrix.
  - $A_1, A_2, A_3, \ldots, A_n$ can be represented by the $n+1$ integers $p_0, p_1, p_2, p_3, \ldots, p_n$

- Let $m(i,j)$ denote the minimum number of multiplications needed to compute $A_i \times A_{i+1} \times \cdots \times A_j$
  - We want to compute $m(1,n)$.

- Divide-and-conquer solution/recursive algorithm:
  - Divide the problem into $j-i$ subproblems by considering the outer-level parentheses in all $j-i$ possible positions. (E.g., $(A_i \times A_{i+1} \times \cdots \times A_k) \times (A_{k+1} \times \cdots \times A_j)$ corresponds to multiplying a $p_{i-1} \times p_k$ matrix by a $p_k \times p_j$ matrix.)

- Recursively find best way of solving subproblems. (i.e., find the best way of computing $A_i \times A_{i+1} \times \cdots \times A_k$ and the best way of computing $A_{k+1} \times A_{k+2} \times \cdots \times A_j$)
- Pick best solution.

- Algorithm expressed in terms of $m(i,j)$:

$$m(i,j) = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m(i,k) + m(k+1,j) + p_{i-1} \cdot p_k \cdot p_j\} & \text{if } i < j \end{cases}$$

- Program:

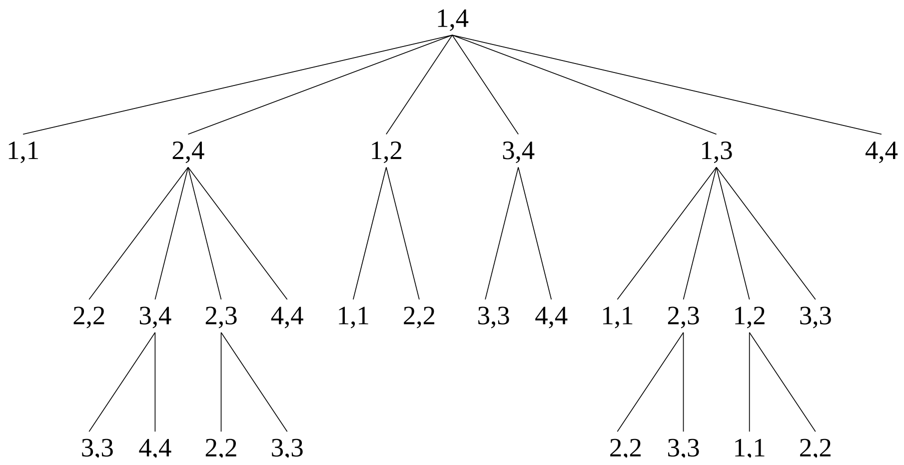
```
MATRIX-CHAIN(i, j)
    IF i = j THEN return 0
    m(i, j) = ∞
    FOR k = i TO j − 1 DO
        q = MATRIX-CHAIN(i, k) + MATRIX-CHAIN(k + 1, j) + p_{i−1} · p_k · p_j
        IF q < m(i, j) THEN m(i, j) = q
    OD
    Return m(i, j)
END MATRIX-CHAIN

Return MATRIX-CHAIN(1, n)
```

- Running time: Let $T(k)$ be the time to compute the optimal computation order for the product of k matrices. We define $T(1) = 0$.

$$\begin{aligned} T(n) &= \sum_{k=1}^{n-1} (T(k) + T(n-k) + c) \\ &= cn + 2 \cdot \sum_{k=1}^{n-1} T(k) \\ &= T(n-1) + c + 2T(n-1) \\ &= 3T(n-1) + c \\ &\sim \frac{3c}{2} 3^{n-2} \end{aligned}$$

- Easy to prove by induction or telescoping the recurrence.

- Problem is that, as in the Fibonacci example, we compute the same result over and over again.
  - Example: Recursion tree for MATRIX-CHAIN$(1,4)$



For example, we compute MATRIX-CHAIN$(3, 4)$ twice.

- Solution is to "remember" values we have already computed in an $n \times n$ table—*memoization*

```
MATRIX-CHAIN(i, j)
    IF i = j THEN return 0
    IF m(i, j) < ∞ THEN return m(i, j)    /* This line has changed */
    FOR k = i to j − 1 DO
        q = MATRIX-CHAIN(i, k) + MATRIX-CHAIN(k + 1, j) + p_{i−1} · p_k · p_j
        IF q < m(i, j) THEN m(i, j) = q
    OD
    return m(i, j)
END MATRIX-CHAIN

FOR i = 1 to n DO
    FOR j = i to n DO
        m(i, j) = ∞
    OD
OD

return MATRIX-CHAIN(1, n)
```

- Running time:
  - $\Theta(n^2)$ different calls to MATRIX-CHAIN(i, j).
  - The first time a call is made it takes $O(n)$ time, *not* counting recursive calls.
  - When a call has been made once it costs $O(1)$ time to make it again.
    $\Longrightarrow \quad O(n^3)$ time
  - Another way of thinking about it: $\Theta(n^2)$ total entries to fill; it takes $O(n)$ time to fill each one.

1.3 Alternative view of Dynamic Programming

- Often (including in the book) dynamic programming is presented in a different way, namely, as filling up a table from the bottom to the top.

- Matrix-chain example:
  Key is that $m(i, j)$ only depends upon $m(i, k)$ and $m(k + 1, j)$ where $i \leq k < j$.

$\Longrightarrow$ If we have already computed $m(i, k)$ and $m(k+1, j)$ for each k, then we can compute $m(i, j)$ in $O(|j - i|)$ time by taking the appropriate minimum over all k.

  - We can easily compute $m(i, i)$ for all $1 \leq i \leq n$: $m(i, i) = 0$
  - Then we can easily compute $m(i, i + 1)$ for all $1 \leq i \leq n - 1$:
    $m(i, i + 1) = m(i, i) + m(i + 1, i + 1) + p_{i-1} \cdot p_i \cdot p_{i+1}$
  - Then we can compute $m(i, i + 2)$ for all $1 \leq i \leq n - 2$:
    $m(i, i+2) = \min\{m(i, i) + m(i+1, i+2) + p_{i-1} \cdot p_i \cdot p_{i+2},\ m(i, i+1) + m(i+2, i+2) + p_{i-1} \cdot p_{i+1} \cdot p_{i+2}\}$

    $\vdots$
  - Until finally we compute $m(1, n)$ and we're done.
  - Computation order is indicated by the numbers in the slots below:

j

i

i \ j	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2		1	2	3	4	5	6
3			1	2	3	4	5
4				1	2	3	4
5					1	2	3
6						1	2
7							1

− Computation order

- Program:

```
FOR i = 1 to n DO
    m(i,i) = 0
OD
FOR l = 1 to n − 1 DO
    FOR i = 1 to n − l DO
        j = i + l
        m(i,j) = ∞
        FOR k = 1 to j − 1 DO
            q = m(i,k) + m(k+1,j) + p_{i−1} · p_k · p_j
            IF q < m(i,j) THEN m(i,j) = q
        OD
    OD
OD
```

- Analysis:
  - $O(n^2)$ entries, $O(n)$ time to compute each $\Longrightarrow O(n^3)$ total time.
- Note:
  - I like recursive (divide-and-conquer) thinking, because you don't need a new idea (and write a totally new program)—just use table lookup!
  - Book seems to like bottom-up method better.

1.4 Overview of dynamic programming

Dynamic programming is a way of improving on inefficient divide-and-conquer algorithms. By "inefficient", we mean that the same recursive call is made over and over.

- If same subproblem is solved several times, we can use table to store result of a subproblem the first time it is computed and thus never have to recompute it again.
- Alternatively, we can think about filling up a table of subproblem solutions from the bottom up.