Topic 20: Shortest Paths in Graphs

(CLRS 24.0–24.3, 25.2)

CPS 230, Fall 2001

Digraph $G = (V, E)$ with weight function $W : E \to \Re$

Weight of path $p = v_1 \to v_2 \to \cdots \to v_k$ is

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$

"Shortest" path = path of minimum weight.

Applications

- Static/dynamic network routing
- Robot motion planning

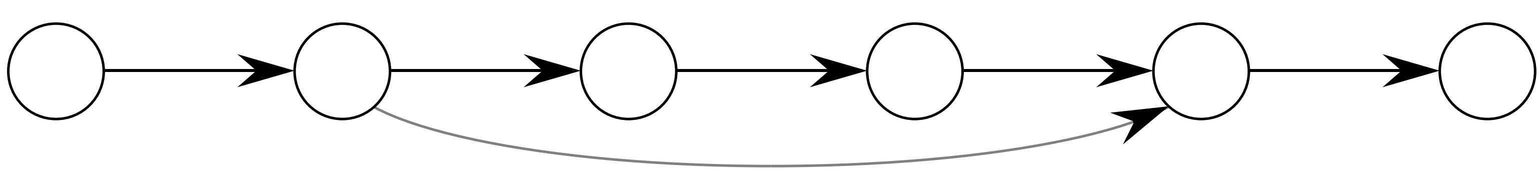
- Different variants of shortest path problem:
  - *Single pair shortest path*: Find shortest path from u to v.
  - *Single source shortest path (SSSP)*: Find shortest path from source s to all vertices $v \in V$.
  - *All pair shortest path (APSP)*: Find shortest path from u to v for all $u, v \in V$.
- Note:
  - No algorithm is known for computing a single pair shortest path better than solving the ("bigger") SSSP problem.
  - APSP can be solved by running SSSP $|V|$ times.

  $\Longrightarrow$ Lets focus on the SSSP problem.

Optimal Substructure

Theorem: Subpaths of shortest paths are shortest paths.

Proof: Cut and paste:



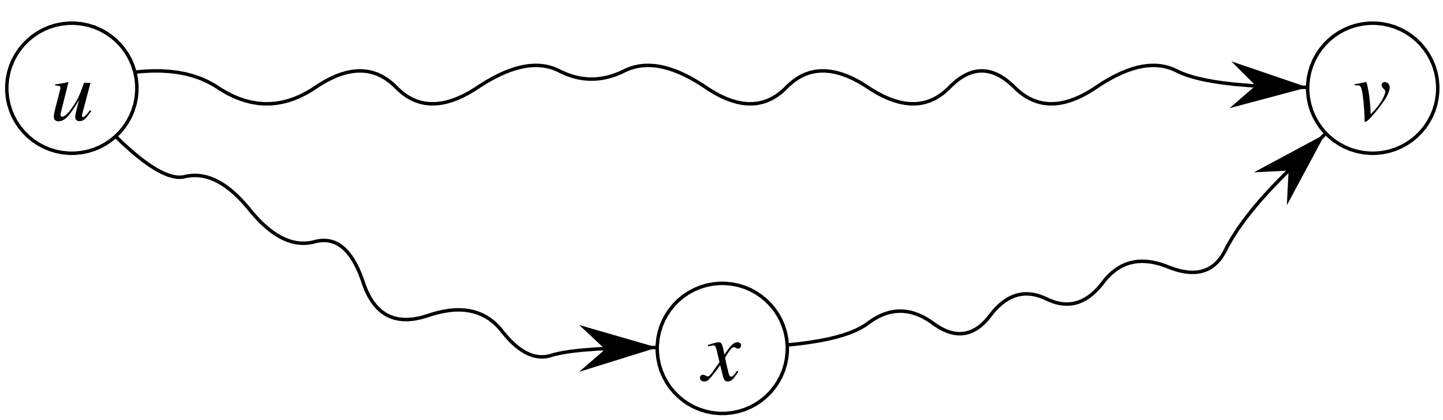
If some subpath were **not** a shortest path,
we could substitute the shorter subpath and create an
even shorter total path. □

Triangle Inequality

Definition: $\delta(u, v) \equiv$ weight of a shortest path from u to v.

Theorem: $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$

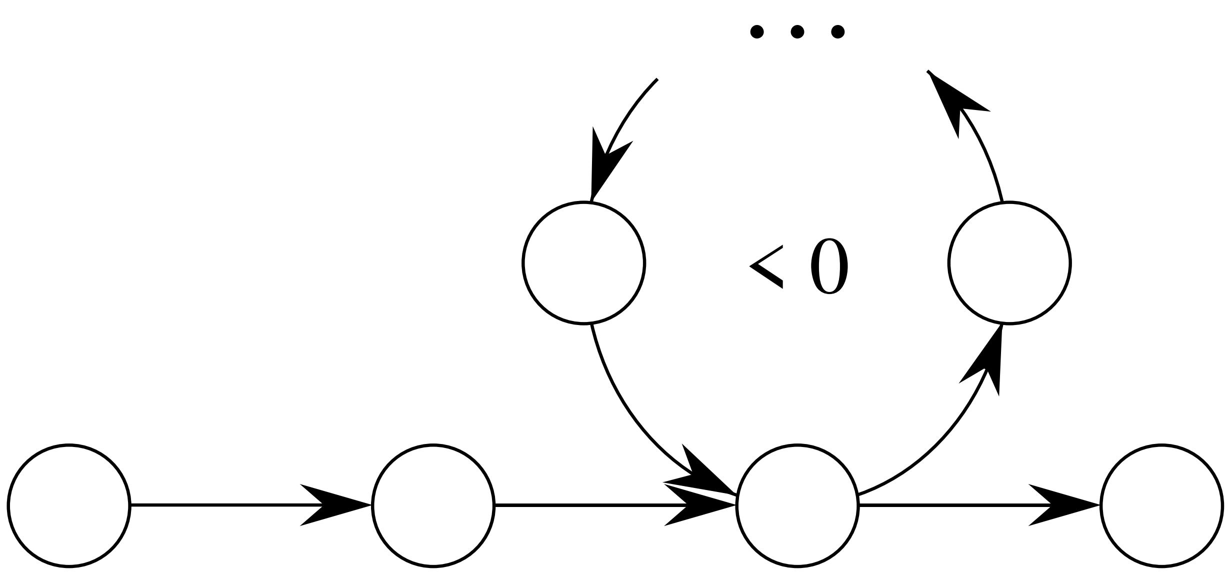
Proof: Shortest path $u \to \cdots \to v$ is no longer than any other path $u \to \cdots \to v$. I.n particular, it is no longer than the path concatenating the shortest path $u \to \cdots \to x$ with the shortest path $x \to \cdots \to v$.
$\square$



Is shortest-path well-defined?

Negative weight cycle $\Rightarrow$ no shortest path.

Argument: Can shorten path by traversing cycle. $\square$



Bellman-Ford Algorithm

Most basic "single-source" shortest-**paths** algorithm

- Finds shortest path weights from specified **source** s to **all** $v \in V$
- Maintains estimate $d[v]$ of path length from s to v, which is updated iteratively
- Actual paths easily reconstructed (CLRS §24.3)

Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
for each v ∈ V
      do d[v] ← ∞
d[s] ← 0    ▷ INITIALIZE-SINGLE-SOURCE(G, s)

for i ← 1 to |V| − 1
      do for each edge (u, v) ∈ E ▷ RELAX
            do if d[v] > d[u] + w(u, v)
                  then d[v] ← d[u] + w(u, v)

for each edge (u, v) ∈ E
      do if d[v] > d[u] + w(u, v)
            then no solution
```

Why call it "Relax"? Chain is getting tighter!
Think of violations of $d[v] \leq d[u] + w(u, v)$ as "pressure". The pressure is relaxed by shortening the estimate of the distance from s to v.

Bellman-Ford Algorithm

Three code sections:

- Lines 1 – 3:
  Initialize: $d[v]$, which will converge to shortest-path values δ.

- Lines 4 – 7:
  Relax: For $|V| - 1$ times, do the following:
  For each edge, do a **relaxation step**.

- Lines 8 – 10:
  Test: Was a solution achieved (iff no negative-weight cycles)?

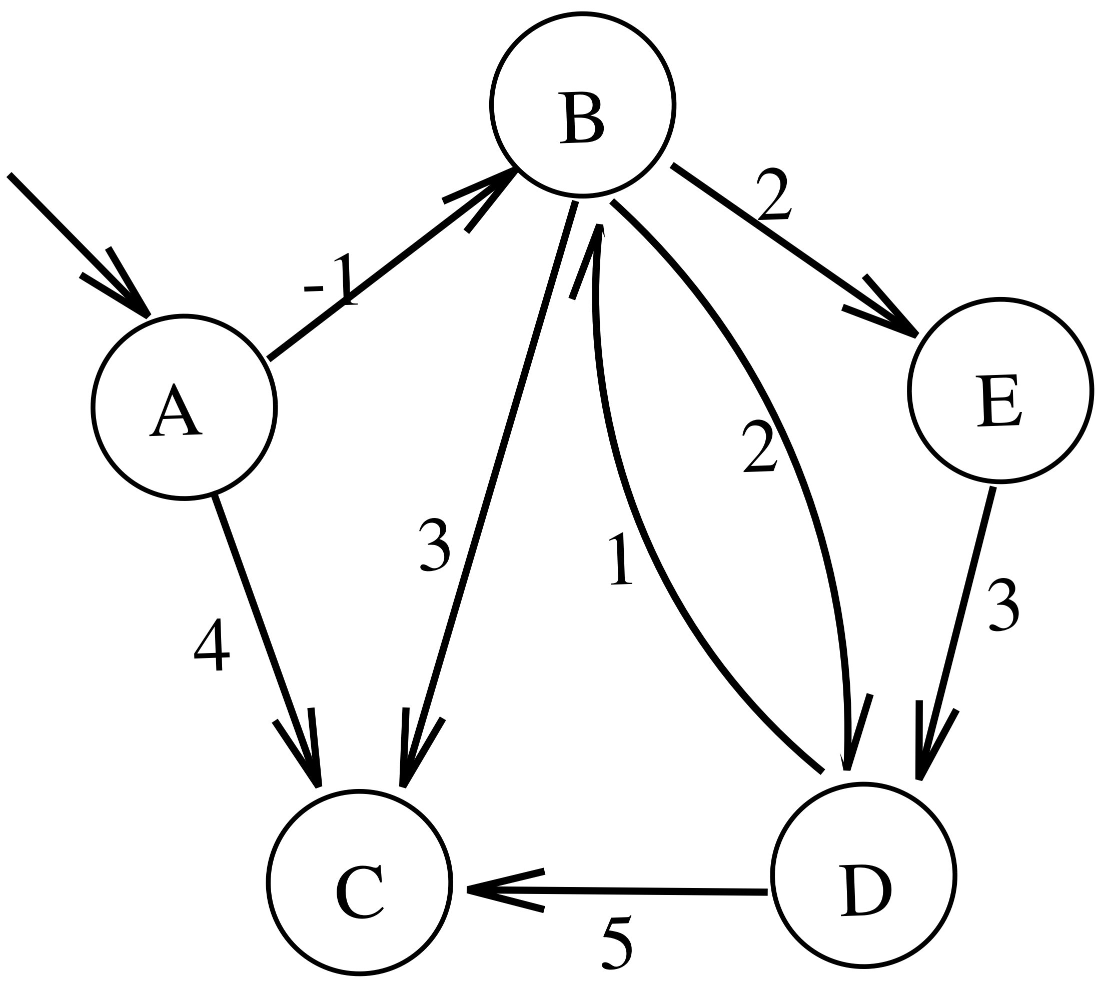
Bellman-Ford Algorithm: Running time

Running Time: $O(V \cdot E)$

- constants are good
- it is simple
- short code

very practical.

Bellman-Ford Algorithm Example



Bellman-Ford Algorithm Example

- **Initialization.** Put initial d values in nodes: $A \leftarrow 0$, rest $\leftarrow \infty$.

- **1st relaxation pass.** Process edges in order (A, B), (A, C), (B, C), (B, D), (D, B), (D, C), (E, D), (B, E).

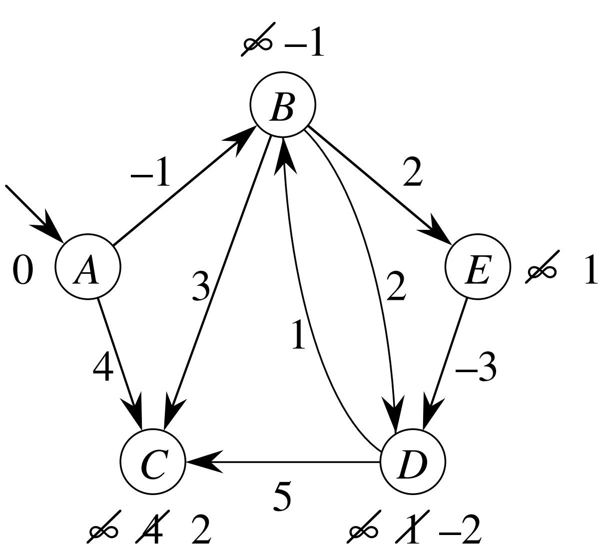
- **2nd relaxation pass.** Process edges in same order. Only D changes.

Bellman-Ford Algorithm Example

- Can stop when no change is detected

d	A	B	C	D	E
init	0	∞	∞	∞	∞
pass 1	0	-1	2	1	1
pass 2	0	-1	2	-2	1
pass 3	0	-1	2	-2	1

- The distances in each pass and the convergence speed of the algorithm depend on the order that the edges are processed.



Bellman-Ford Algorithm: Lemma

Lemma: $d[v] \geq \delta(s,v)$ always.

Proof:

- Initially true
- Let v be first vertex for which $d[v] < \delta(s,v)$, and let u be vertex that caused $d[v]$ to change:

$$d[v] = d[u] + w(u,v)$$

- Then

$$\begin{aligned} d[v] &< \delta(s,v) \\ &\leq \delta(s,u) + \delta(u,v) \quad \text{(Triangle inequality)} \\ &\leq \delta(s,u) + w(u,v) \quad \text{(shortest path} \leq \text{specific path)} \\ &\leq d[u] + w(u,v) \qquad (v \text{ is first violation}) \end{aligned}$$

contradicts $d[v] = d[u] + w(u,v)$ (above).

Therefore, once $d[v]$ reaches $\delta(s,v)$, it can't change (since $d[v]$ can only decrease, never increase).

Bellman-Ford Algorithm: Correctness

Claim: Bellman-Ford correct (i.e., after $|V|-1$ passes, all the d values are correct)

Proof: Let v be any vertex, and consider a shortest path from s to v (assuming no neg-weight cycles):

$$s \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v$$

- Initially, $d[s] = 0$ is correct (and doesn't change thereafter since the algorithm never increases d)

Bellman-Ford Algorithm: Correctness

Proof: *(continued)*

- After 1 pass through edges, $d[v_1]$ is correct (and doesn't change thereafter)

  ($d[s]$ is correct, and by the optimal substructure, the shortest distance is $w(s, v_1)$.
  1st pass sets $d[v_1] \leftarrow d[s] + w(s, v_1)$, which is the right answer.)

- After 2 passes through edges, $d[v_2]$ is correct (and doesn't change thereafter)

$$\vdots$$

- Terminates in $|V| - 1$ passes. Why?
  If no negative-weight cycles:
  – every shortest path is **simple** (no cycles)
  – longest simple path has $|V| - 1$ edges

Bellman-Ford Algorithm: Correctness

Proof: *(continued)*

- Thus if no neg-weight cycles, all the $d[v]$ converge in $|V|-1$ passes.
  Equivalently, if a value $d[v]$ fails to converge after $|V|-1$ passes, $\exists$ neg-weight cycle.
- Last part of algorithm tests for success by seeing if another pass would change anything.

The converse is also true:
If $\exists$ neg-weight cycle reachable from s, then some value $d[v]$ fails to converge after $|V|-1$ passes.

(Proof left as exercise.) *(CLRS Theorem 24.4.)*

So... Bellman-Ford can be used to check for negative-weight cycles.

SSSP in DAG

- If graph is acyclic, we can solve SSSP by relaxing outgoing edges from vertices in the topological sort order of the vertices.
- Running time is $O(|E|)$.

Dijkstra's Algorithm

Dijkstra's Algorithm:

- Non-negative edge weights
  $\Longrightarrow$ shortest paths always exist.
  (If there are no negative weights, Dijkstra's algorithm is faster than Bellman-Ford.)
- Like breadth-first-search
  (If all weights = 1, use BFS, otherwise Dijkstra.)
- Use for Q a priority queue keyed by $d[v]$.
  Greedy, like Prim's algorithm for MST
  BFS used FIFO queue

Dijkstra's Algorithm: Pseudocode

$\textsc{Dijkstra}(G, w, s)$

1 **for** each $v \in V$
2   **do** $d[v] \leftarrow \infty$
3 $d[s] \leftarrow 0$
4 $S \leftarrow \emptyset$
5 $Q \leftarrow V$
6 **while** $Q \neq \emptyset$
7   **do** $u \leftarrow \textsc{Extract-Min}(Q)$
8         $S \leftarrow S \cup \{u\}$
9   **for** each $v \in Adj[u]$
10   **do if** $d[v] > d[u] + w(u, v)$
11   **then** $d[v] \leftarrow d[u] + w(u, v)$

What is line 7 doing?
What is line 11 doing?

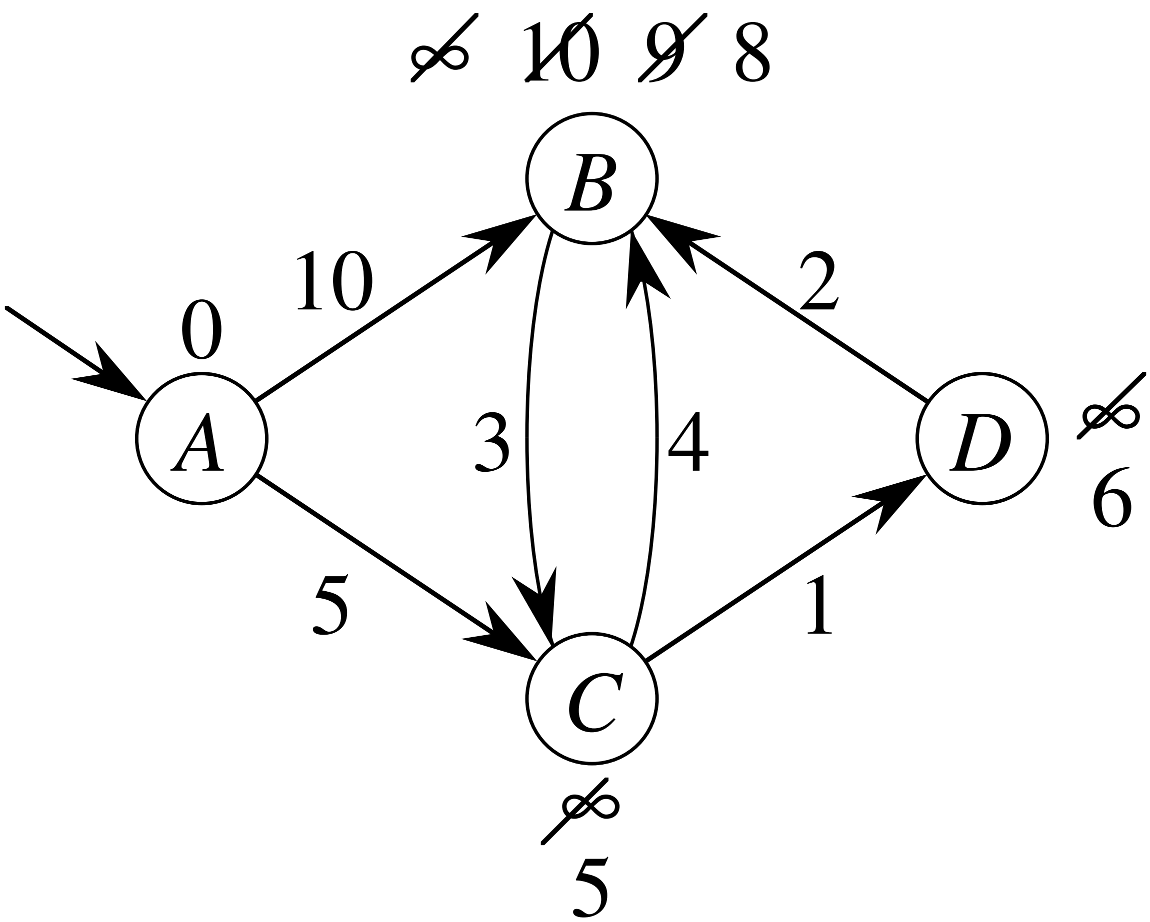
Dijkstra's Algorithm: Notes

Observe:

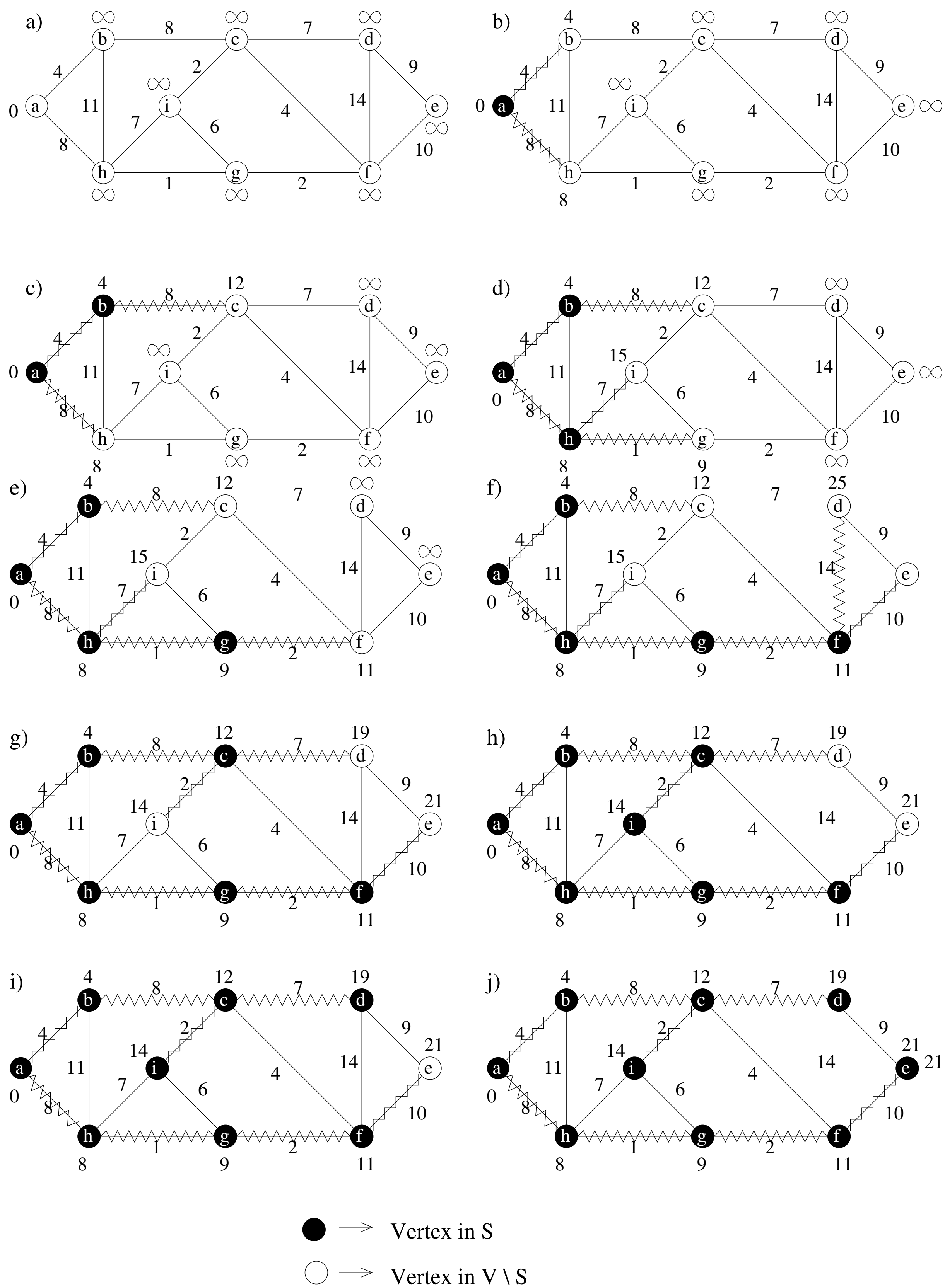
- relaxation step
- setting $d[v]$ updates Q
  (DECREASE-KEY operation)
- similar to Prim's minimum-spanning-tree algorithm

Dijkstra's Algorithm

Example:



Another Example of Dijkstra's Algorithm



Dijkstra's Algorithm: Run-Time Analysis

- $\textsc{Extract-Min}$ executed $|V|$ times
- $\textsc{Decrease-Key}$ executed $|E|$ times

$$\text{Time} = |V| \cdot T_{\textsc{Extract-Min}} + |E| \cdot T_{\textsc{Decrease-Key}}$$

Analysis: Look at different Q implementations.

Priority queue	$T_{\textsc{Extract-Min}}$	$T_{\textsc{Decrease-Key}}$	Total
array	$O(\lvert V\rvert)$	$O(1)$	$O(\lvert V\rvert^2)$
binary heap	$O(\log \lvert V\rvert)$	$O(\log \lvert V\rvert)$	$O(\lvert E\rvert \log \lvert V\rvert)$
Fibonacci heap	$O(\log \lvert V\rvert)$	$O(1)$	$O(\lvert V\rvert \log \lvert V\rvert + \lvert E\rvert)$
	amortized	amortized	worst case

Dijkstra's Algorithm: Analysis

- Q = unsorted array:

  scan to find minimum
  just index and update to change key

- Q = Fibonacci heap

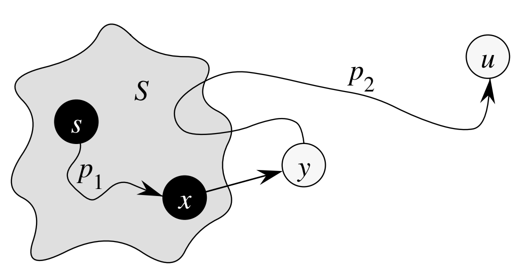
  Note advantage of amortized analysis:
  Can use amortized Fibonacci heap bounds in analysis, as if they were worst-case bounds, and get (real) worst-case bounds on aggregate running time.

Dijkstra's Algorithm: Correctness

Correctness: Prove that whenever u is added to S, $d[u] = \delta(s, u)$

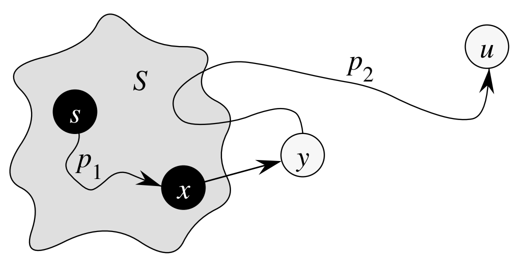
Proof :

- Note that $\forall\ v\ \ d[v] \geq \delta(s, v)$
- Let u be first vertex added to S such that there is a path from s to u of length shorter than $d[u]$ $\implies\quad d[u] > \delta(s, u)$
- Let's consider the set S immediately before u is processed (i.e., u is not yet in S, but is about to be picked next):



Dijkstra's Algorithm: Correctness

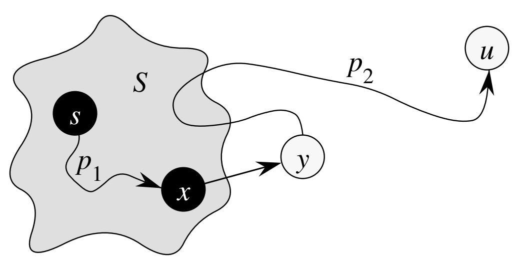
(Proof continued)



- Let y be first vertex $\in V - S$ on actual shortest path from s to u
  $\Longrightarrow \quad d[y] = \delta(s, y)$
  Because:
  - $d[x]$ is set correctly for y's predecessor $x \in S$ on the shortest path (by choice of u as first choice for which that's not true)
  - When we put x into S, we relaxed (x, y), giving $d[y]$ its correct value

Dijkstra's Algorithm: Correctness

(Proof continued)



$$\begin{aligned} d[u] &> \delta(s,u) \\ &= \delta(s,y) + \delta(y,u) \qquad \text{(optimal substructure)} \\ &= d[y] + \delta(y,u) \\ &\geq d[y] \qquad \text{(no negative weights)} \end{aligned}$$

- But $d[u] > d[y] \implies$ algorithm would have chosen y to process next, not u. *Contradiction.*

All pairs shortest path (APSP) with nonnegative weights

- In the APSP problem, we want to compute the shortest path between any two vertices $u, v \in V$.
  - The output has size $O(|V|^2)$, so we cannot hope to design a better than $O(|V|^2)$-time algorithm.
- We can solve the problem simply by running Dijkstra's algorithm $|V|$ times $\Longrightarrow$ $O(|V| \cdot |E| \log |V|)$ algorithm.
  - In the worst case (dense graph) the running time is $O(|V|^3 \log |V|)$.
- The Floyd-Warshall algorithm (see CLRS Exercise 25.2-4) runs in only $O(|V|^3)$ time by working on adjacency matrix A:

```
FOR k = 1 to |V| do
    FOR i = 1 to |V| DO
        FOR j = 1 to |V| DO
            IF  A[i,j] > A[i,k] + A[k,j]
        THEN
            A[i,j] = A[i,k] + A[k,j]
          FI
        OD
    OD
OD
```

- Correctness:
  - We prove correctness by induction.
  - We will prove that, after each iteration of the outer loop on k, the following *invariant* holds: After the kth iteration (out of $|V|$), $A[i, j]$ contains the length of the shortest path from v_i to v_j that (apart from v_i and v_j) only contains vertices of index at most k.
    $\Longrightarrow$ When algorithm terminates we have solved APSP.
  - Proof:
    * Invariant holds initially for $k = 0$ (i.e., we start with adjacency matrix A, and the only allowed path from v_i to v_j is the edge from v_i to v_j).
    * When "adding" vertex with index k, we explicitly check all new paths between v_i and v_j that pass through v_k, for all $|V|^2$ pairs of v_i and v_j.
- Note:
  - We can easily produce adjacency-matrix from adjacency list in $O(|V^2|)$ time.
  - Algorithm runs in $O(|V|^3)$ time, even if the graph is sparse. Using algorithm based on Dijkstra's algorithm we will get much better performance for sparse graphs.
  - Using a Fibonacci heap, in which DECREASE-KEY operations take constant time amortized, Dijkstra's algorithm can be improved to $O(|V|^2 \log |V| + |V| \cdot |E|) = O(|V|^3)$ time.