Algorithms
Professor John Reif

ALG 4.2
Universal Hash Functions:

CLR - Chapter 34
Auxillary Reading Selections:
AHU-Data Section 4.7
BB Section 8.4.4
Handout: Carter & Wegman, "Universal Classes of Hash Functions", JCSS, Vol. 18, pp. 143-154, 1979.

Hash Function

$f : A \longrightarrow B$

keys (A) indices (B)

f has *conflict* at x,y ε A if
x≠y but f(x) = f(y)

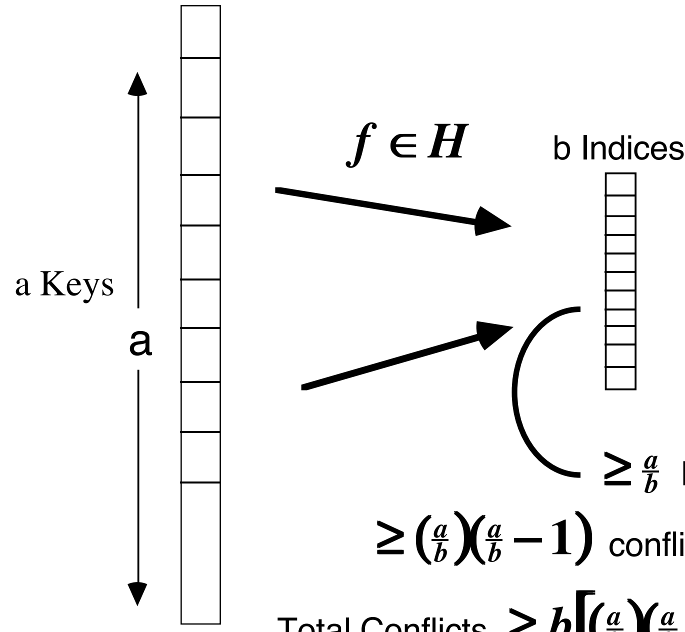
$$\sigma_f(x, y) = \begin{cases} 1 & \text{if } x \neq y \text{ and } f(x) = f(y) \\ 0 & \text{else} \end{cases}$$

If H is a set of hash functions,

$$\sigma_H(x,y) = \sum_{f \varepsilon H} \sigma_f(x,y)$$

for set of keys S,

$$\sigma_H(x,S) = \sum_{f \varepsilon H} \sum_{y \varepsilon S} \sigma_f(x,y)$$



$\geq \frac{a}{b}$ keys / index

$\geq \left(\frac{a}{b}\right)\left(\frac{a}{b} - 1\right)$ conflicts / index

Total Conflicts $\geq b\left[\left(\frac{a}{b}\right)\left(\frac{a}{b} - 1\right)\right]$

$\geq \frac{a^2}{b} - a$

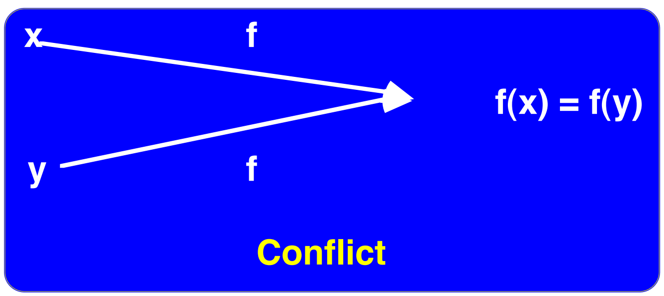
H is a *universal* $_2$ set of hash functions

if $\sigma_H(x,y) \le \frac{|H|}{|B|}$ for all $x,y \in A$

i.e. no pair of keys x,y are mapped

into the same index by $> \frac{1}{|B|}$

of all functions in H



Proposition 1

Given *any* set H of hash fn,

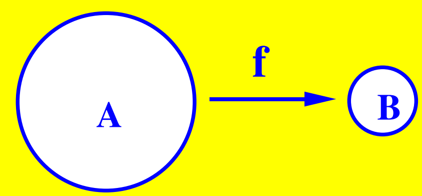
$\exists$ x,y $\in$ A s.t.

$$\sigma_H(x,y) > |H| \left(\frac{1}{|B|} - \frac{1}{|A|} \right)$$

proof

let $a = |A|$, $b = |B|$

By counting, we can show



$$\sigma_f(A,A) \ge b\left(\tfrac{a}{b}-1\right)^2 \ge \tfrac{a^2}{b} - a$$

Thus

$$\sigma_H(A,A) \geq a^2 \, |H| \left(\frac{1}{b} - \frac{1}{a}\right)$$

By the pidgeon hole principle

$\exists$ x,y ε A s.t

$$\sigma_H(x,y) \geq |H| \left(\frac{1}{b} - \frac{1}{a}\right)$$

note

in most applications, $|A| >> |B|$

and then any *universal* $_2$ *class has*

asymptotically a *minimum number of conflicts*

Proposition 2: Let $x \in A$, $S \subseteq A$

For f chosen randomly from a universal $_2$ class H of hash functions, the expected number of colisions is

$$\sigma_f(x,S) \leq \frac{|S|}{|B|}$$

proof

$$E(\sigma_f(x,S)) = \frac{1}{|H|} \sum_{f \in H} \sigma_f(x,S)$$

$$= \frac{1}{|H|} \sum_{y \in S} \sigma_H(x,y) \quad \text{by definition}$$

$$\leq \frac{1}{|H|} \sum_{y \in S} \frac{|H|}{|B|} \quad \text{by definition of universal}_2$$

$$= \frac{|S|}{|B|}$$

application

associative memory storage of |S| keys onto |B| linked lists.

Given key $x \in A$, store x in list f(x)

Proposition 2 implies each list has expected

$$length \leq \frac{|S|}{|B|} = 0(1) \text{ if } |B| \geq |S|$$

Gives 0(1) time for STORE, RETRIEVE, and DELETE operations

Proposition 3

Let R be a sequence of requests with k insertion operations into an associative memory.

If f is chosen at random from set of universal$_2$ class H, the expected

total cost of all k searches is

$$\leq |R| \left(1 + \frac{k}{|B|}\right).$$

proof

There are |R| total search ops, and each takes by Proposition 2 expected

$$\text{time} \leq 1 + \frac{k}{|B|}.$$

note

if $|B| \geq k$, then expected total time is O(|R|).

Bounds on distribution of $\sigma_f(x,S)$

Proposition 4 Let $x \in A$, $S \subset A$

Let μ = expected value of $\sigma_f(x,S)$
For f chosen randomly from universal $_2$ set of functions H,

$$\text{Prob}\ (\sigma_f\ (x,S) > t \cdot \mu) < \frac{1}{t}$$

proof

immediate from Markov bound

improved **bounds on probability:**

$$\text{prob} \le \frac{11}{t^4} \quad \text{for universal hash fns. } H_2,\ H_3$$

(using 2nd and 4th moments of prob. distribution.)

H = universal$_2$ set of hash functions.

E_1= Expected cost of *random* set of k requests using a *worst case* function f in H *(random input)*

E_2= Expected cost of *worst case* set of k requests using a *random* function f in H *(randomized algorithm)*

Prop 5 $E_1 \geq (1-\varepsilon) E_2$ where $\varepsilon = \frac{|B|}{|A|}$

proof

Let $a = |A|$, $b = |B|$.

Prop 2 implies $E_2 \leq 1 + \frac{|S|}{b}$

Suppose S is chosen randomly. for x, y ε S,

$$E(\sigma_f(x,y)) = \frac{1}{a^2} \sigma_f(A,A)$$

$$\geq \frac{1}{a^2} \left| a^2 \left(\frac{1}{b} - \frac{1}{a} \right) \right| \text{ by Prop 1}$$

$$\geq \left(\frac{1}{b} - \frac{1}{a} \right)$$

So $E_1 \geq 1 + E(\sigma_f(x,S))$

$$\geq 1 + |S| \left(\frac{1}{b} - \frac{1}{a} \right)$$

Example of Universal $_2$ ***Class***

Set of Keys Table

Let $A = \{0,1, \ldots, a-1\}$ Set of Keys
$B = \{0,1, \ldots, b-1\}$ Table

Let p be a prime $\geq a$

$Z_p = \{0,1, \ldots, p-1\}$ = number field mod p

define $g : Z_p \to B$ s.t.
$g(x) = x \bmod b$

define for $n, m \in Z_p$ with $m \neq 0$,
$h_{n,m}: A \to Z_p$
with $h_{n,m}(x) = (mx+n) \bmod p$

define $f_{n,m}: A \to B$ s.t. $f_{n,m}(x) = g(h_{m,n}(x))$

$$H_1 = \{ f_{m,n} \mid m, n \in Z_p, \; m \neq 0 \}$$

Claim: H_1 is universal $_2$

Lemma

for distinct $x, y \in A$,

$\sigma_{H_1}(x,y) = \sigma_g(Z_p, Z_p)$

proof

$\sigma_g(Z_p, Z_p) = |\{(r,s) \mid r,s \in Z_p, r \neq s, g(r) = g(s)\}|$

Observe that the linear equations:

$xm + n = r \pmod p$

$ym + n = s \pmod p$

have *unique* solutions in Z_p

So $(r,s) = (h_{m,n}(x), h_{m,n}(y))$ then

$(f_{m,n}(x) = f_{m,n}(y)$ if and only if $g(r) = g(s))$

$\sigma_H(x,y)$ is the number of such pairs in

$(r,s) \in \sigma_g(Z_p, Z_p)$

Theorem

H_1 is universal$_2$

proof Let $n_i = |\{t \in Z_p \mid g(t) = i\}|$

By definition of $g(x) = x \bmod b$,

$$\Rightarrow \quad n_i \leq \frac{p-1}{b} + 1$$

For any given r, the number of s where $s \neq r$ and $g(r) = g(s)$ is

$$\sigma_g(r, Z_p) \leq \frac{p-1}{b}.$$

But there are p choices of r,

so $p \cdot \left(\frac{(p-1)}{b}\right) \geq \sigma_g(Z_p, Z_p)$

$= \sigma_{H_1}(x,y)$ by Lemma

(Also note $\sigma_H(x,x) = 0$)

Hence $\sigma_{H_1}(x,y) \leq \frac{|H_1|}{b}$ since $|H_1| = p(p-1)$

so H_1 is universal$_2$

Universal Hash Fns on *Long* keys

Given class of hash functions H,
define hash functions $J = \{h_{f,g} \mid f,g \in H\}$

where $h_{f,g}(x_1, x_2) = f(x_1) \oplus g(x_2)$

$\uparrow$ exclusive or

Theorem Suppose $B = \{0,1,\ldots,b=1\}$ where b is a power of 2. Suppose this class of fns $A \to B$

$\exists$ real $r \forall i \varepsilon B \forall x_1, y_1 \varepsilon A,\ x_1 \neq y_1$
$\Rightarrow \left|\{f \varepsilon H \mid f(x_1) \oplus f(y_1) = i\}\right| \leq r|H|$

Then $\forall x, y \in (A \times A),\ x \neq y$
$\left|\{h \varepsilon J \mid h(x) \oplus h(y) = i\}\right| \leq r|H|$

Proof for $x = (x_1, x_2),\ y = (y_1, y_2)$ in $A \times A$

$i \varepsilon B$ then $\left|\{h \varepsilon J \mid h(x) \oplus h(y) = i\}\right|$

$= \left|\{f, g \varepsilon H \mid f(x_1) \oplus g(x_2) \oplus f(y_1) \oplus g(y_2) = i\}\right|$

$= \sum_{y \varepsilon H} \left|\{f \varepsilon H \mid f(x_1) \oplus f(y_1) = i \oplus g(x_2) \oplus g(y_2)\}\right|$

$\leq \left|\{f \in H \mid f(x_1) \oplus f(y_1) = i\}\right| \leq r|H|$

example H_1 with $m = 0$ gives J with $r = \dfrac{1}{|B|}$ universal!

Universal$_2$ Hashing *with out Multiplication*

A = set of d digit numbers base α so, $|A| = \alpha^d$

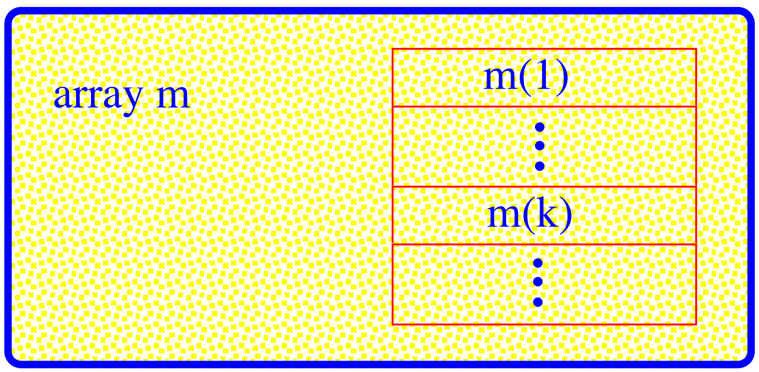
B = set of binary numbers length j

M = arrays of length $d \cdot \alpha$, with elements in B

$\forall m \;\varepsilon\; M$ let $m(k)$ = kth element of array m

$\forall x \;\varepsilon\; A$ let x_k = kth digit of x base α

definition $f_m(x) = m(x_1+1) \oplus m(x_1+x_2+2) \oplus \ldots \oplus m\left(\sum_{k=1}^{d} x_k + k\right)$



Theorem

$H_2 = \{ f_m \mid m \;\varepsilon\; M \}$ is universal$_2$

proof for $x, y \;\varepsilon\; A$,

let $f_m(x) = r_1 \oplus r_2 \oplus \ldots \oplus r_s$ rows of m

$f_m(y) = r_{s+1} \oplus \ldots \oplus r_t$

Then $f_m(x) = f_m(y)$ iff $r_1 \oplus \ldots \oplus r_t = \bar{0}$

But if $x \neq y \Rightarrow \exists\, k$ s.t. r_k in only one of $f_m(x)$, $f_m(y)$

so $\left(f_m(x) = f_m(y) \text{ iff } r_k = \bigoplus_{i \neq k} r_i \right)$

But there are only $|B|$ possibilities for row r_k

so x, y will collide for $\frac{1}{|B|}$ of fns $f_m \;\varepsilon\; H_2$

Hence H_2 is universal$_2$

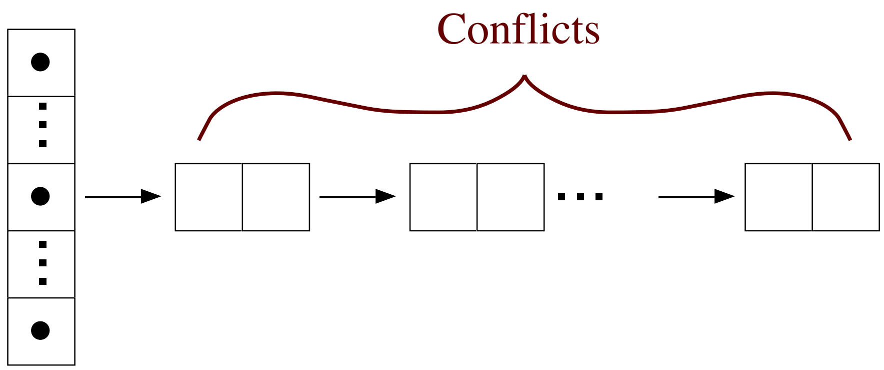
Analysis of Hashing

for Uniform Random Hash fn

load factor $\alpha = \dfrac{\text{\# of keys hashed}}{\text{\# of indicies in Hash Table}}$

Hashing with Chaining

keep list of conflicts at each index



length is ***binomial*** variable

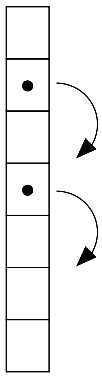
expected length = α

Expected Time Cost per hash = $O(1+\alpha)$

By Chernoff Bounds, with high likelyhood
time cost per hash $\le O(\alpha \log(\#\ \text{keys}))$

Open Address Hashing
(With Uniform Random Hash fn)

Resolve conflicts by applying another hash function



α = load factor = prob. of occupied hash address

rehashes as geometric variable

expected hash time = $\frac{1}{1-\alpha} = 1+\alpha+\alpha^2+\ldots$