#P and the Permanent

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In this lecture, we will prove that the problem of finding the permanent of a 0-1 matrix is #P-complete. Given an $n \times n$, 0-1 matrix A, it can be viewed as an adjacency matrix of a directed graph G on n vertices (with possibly self-loops). It is easy to see that the permanent of A is the number of cycle-covers of G. (A cycle-cover is a sub-graph consisting of a union of disjoint cycles that cover the vertices of G).

The hardness of 0-1PERM is established by showing that the problem of finding the number of cycle-covers of G is hard.

Theorem 6.1 (Valiant). 0-1PERM *is #P-complete.*

Proof. For a directed graph G, a *cycle-cover of* G is a union of simple cycles of G that contains each vertex precisely once. For a weighted, directed graph G, with weight matrix A, we can also view

$$\text{PERM}(G) = \text{PERM}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

as the total weight of all cycle-covers of G, where the weight of a cycle-cover is the product of the weights of all its edges. This interpretation corresponds naturally to the representation of a permutation σ as a union of directed cycles. For example, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 2 & 1 & 6 \end{pmatrix} \in S_6$ then σ can also be written in cycle form as $(1\ 3\ 5)(2\ 4)(6)$ where the notation implies that each number in the group maps to the next and the last maps to the first. (See Figure 6.1.) Thus, for an unweighted graph G, $\text{PERM}(G)$ is the number of cycle-covers of G.

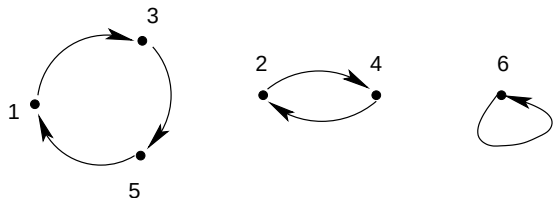

Figure 6.1: Directed graph corresponding to $(1\ 3\ 5)(2\ 4)(6)$

Theorem 6.1 (Valiant). 0-1PERM *is* #P*-complete*.

Proof Idea: We will reduce #3-SAT to 0-1PERM in two steps. Given any 3-SAT formula φ, in the first step, we will create a weighted directed graph G' (with small weights) such that

$$\text{PERM}(G') = 4^{3m} \cdot \#(\varphi)$$

where m is the number of clauses in φ. In second step, we will convert G' to an unweighted graph G such that $\text{PERM}(G') = (\text{PERM}(G) \bmod M)$, where M will only have polynomially many bits.

First, we will construct G' from φ. The construction will be via gadgets. The VARIABLE gadget is shown in Figure 6.2. All the edges have unit weights. Notice that it contains one dotted edge for every occurrence of the variable in φ. Each dotted edge will be replaced by a subgraph which will be described later. Any cycle-cover either contains all dotted edges corresponding to a positive occurrence (and all self-loops corresponding to negative occurrence) or vice versa.

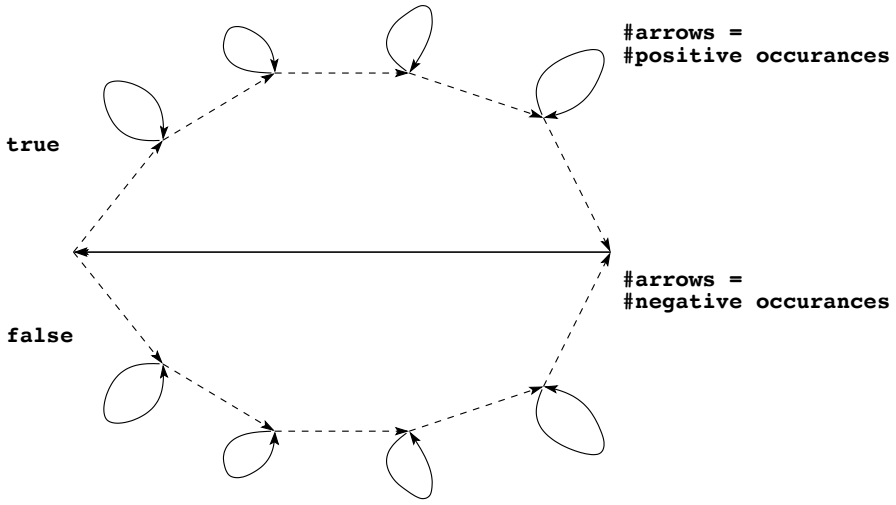

Figure 6.2: The VARIABLE gadget

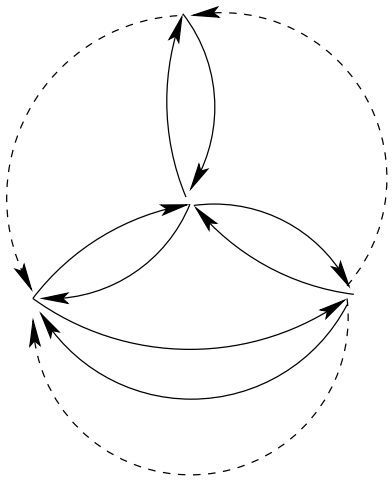

Figure 6.3: The CLAUSE gadget

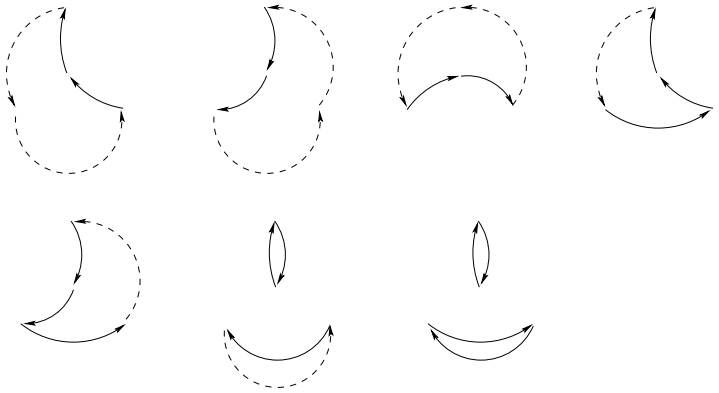

Figure 6.4: The cycle-covers of the CLAUSE gadget

The CLAUSE gadget is shown in Figure 6.3. It contains three dotted edges corresponding to three variables that occur in that clause. All the edges have unit weights. This gadget has the property that

1. in any cycle-cover, at least one of the dotted edges is not used, and
2. for any non-empty subset S of the dotted edges there is precisely one cycle-cover of the gadget that includes all dotted edges but those in S. (See Figure 6.4.)

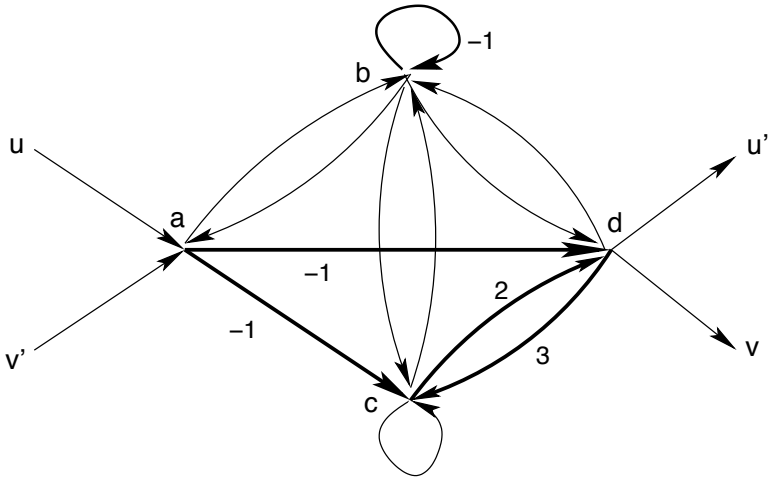

Figure 6.5: The XOR gadget

Now, given any clause C and any literal x contained in it, there is a dotted edge (u, u') in the CLAUSE gadget for the literal and a dotted edge (v, v') in the appropriate side of VARIABLE gadget for the clause. These two dotted edges are replaced by an XOR gadget shown in Figure 6.5.

The XOR gadget has the property that the total contribution of all cycle-covers using none or both of (u, u') and (v, v') is 0. For cycle-covers using exactly one of the two, the gadget contributes a factor of 4. To see this, lets consider all possibilities:

1. None of the external edges is present: The cycle-covers are $(a\ c\ d\ b)$, $(a\ b)(c\ d)$, $(a\ d\ b)(c)$ and $(a\ d\ c\ b)$. The net contribution is (-2) + 6+ (-1) + (-3) = 0.

2. Precisely (u, a) and (a, v') are present: The cycle-covers are $(b\ c\ d)$, $(b\ d\ c)$, $(c\ d)(b)$ and $(c)(b\ d)$. The net contribution is (2) + (3)+ (-6) + (1) = 0.

3. Precisely (v, d) and (d, u') are present: The cycle-covers are $(a\ b)(c)$ and $(a\ c\ b)$. The net contribution is 1 + (-1) = 0.

4. All four external edges are present: The cycle-covers are $(b\ c)$ and $(b)(c)$. The net contribution is 1 + (-1) = 0.

5. Precisely (v, d) and (a, v') are present: In this case the gadget contains a path from d to a (represented with square brackets) as well as a cycle-cover involving the remaining vertices. The contributions to the cycle-covers are $[d\ b\ a](c)$ and $[d\ c\ b\ a]$. The net contribution is 1 + 3 = 4.

6. Precisely (u, a) and (d, u') are present: The cycle-covers are $[a\ d](b\ c)$, $[a\ d](b)(c)$, $[a\ b\ d](c)$, $[a\ c\ d](b)$, $[a\ b\ c\ d]$ and $[a\ c\ b\ d]$. The net contribution is (-1) + 1 + 1 + 2 + 2 + (-1) = 4.

There are $3m$ XOR gadgets. As a result, every satisfying assignment of truth values to φ will contribute 4^{3m} to the cycle-cover and every other assignment will contribute 0. Hence,

$$\text{PERM}(G') = 4^{3m}\#(\varphi)$$

Now, we will convert G' to an unweighted graph G. Observe that $\text{PERM}(G') \leq 4^{3m}2^n \leq 2^{6m+n}$. Let $N = 6m + n$ and $M = 2^N + 1$. Replace the weighted edges in G' with a set of unweighted edges as shown in Figure 6. For weights 2 and 3, the conversion does not affect the total weight of cycle-covers. For weight -1, the conversion blows up the total weight by $2^N \equiv -1(modM)$. As a result, if G is the resulting unweighted graph, $\text{PERM}(G') \equiv \text{PERM}(G) \pmod{M}$.

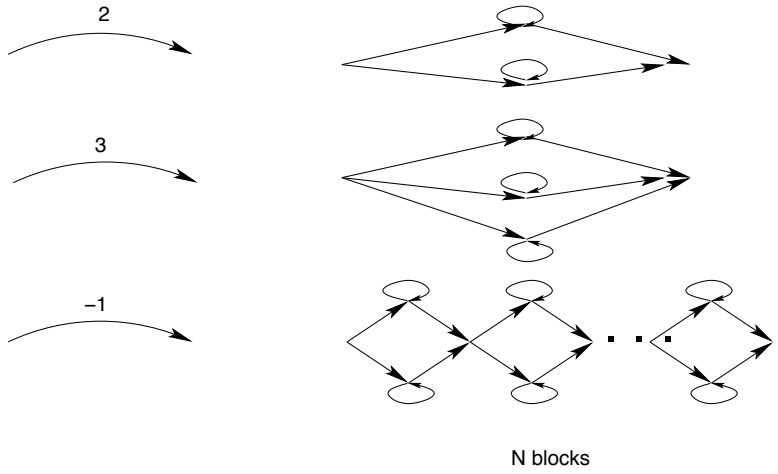

Figure 6.6: Conversion to an unweighted graph

Thus, we have shown a reduction of #3-SAT to 0-1PERM. This proves the theorem.

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