Lecture 5-1. The Incompressibility Method, continued

- We give a few more examples using the incompressibility method. We avoid ones with difficult and long proofs, only give short and clean ones to demonstrate the ideas.
- These include:
  - Tournaments
  - Fast Adders for random input
  - Coin-weighing problem
  - Turing Machines with Bounded numbers of Heads
  - Pushdown Machines
- Then we will survey the ideas of the solutions of some major open questions. They have difficult proofs, but it is sufficient show you just the ideas.

Fast adder

- Example. Fast addition on average.
  - Ripple-carry adder: n steps adding n-bit numbers.
  - Carry-lookahead adder: 2 log n steps.
  - Burks-Goldstine-von Neumann (1946): logn expected steps.

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S= x⊕y; C= carry sequence;
while (C≠0) {
    S= S⊕C;
    C= new carry sequence; }
```

Average case analysis: Fix x, take random y s.t. $C(y|x) \geq |y|$

x = … u1 …
y = … û1 …,

(Max such u is carry length)
û is complement of u

If $|u| > \log n$, then $C(y|x)<|y|$. Average over all y, get logn. QED

Coin weighing problem

- A family $D=\{D_1,D_2, \ldots , D_j\}$ if subsets of $N=\{1,\ldots,n\}$ is called a distinguishing family for N, if for any two distinct subsets M and M' of N there exists an i $(1 \leq i \leq j)$ s.t. $|D_i \cap M|$ is different from $|D_i \cap M'|$.
- Let f(n) denote the minimum of |D| over all distinguishing families for N.
- To determine f(n) is known as coin-weighting problem.
- Erdos, Renyi, Moser, Pippenger:

$$f(n) \geq (2n/\log n)[1+O(\log\log n/\log n)]$$

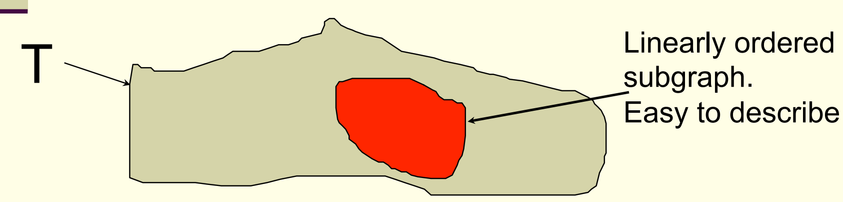
Combinatorics

- Example There is a tournament (complete directed graph) T of n players that contains no large transitive subtournaments (>1 + 2 log n).

Proof by Picture: Choose a random T.

- One bit codes an edge. $C(T) \geq n(n-1)/2$.
- If there is a large transitive subtournament, then a large number of edges are given for free!

C(T)< n(n - 1)/2 - subgraph-edges + overhead ■



Theorem: $f(n) \geq (2n/\log n)[1+O(\log\log n/\log n)]$

Proof. Choose M such that

$C(M|D) \geq n$.

Let $d_i = |D_i|$ and $m_i = |D_i \cap M|$.

Value m_i is within the range $d_i / 2 \pm O(\sqrt{d_i} \log i)$. Therefore, given d_i, each m_i can be described by its discrepancy with $d_i/2$, with gives

$$C(m_i|D_i) \leq \tfrac{1}{2} \log d_i + O(\log\log i)$$
$$\leq \tfrac{1}{2} \log n + O(\log\log n)$$

Since D is a distinguishing family for N, given D, the values of $m_1, \ldots, m_j$ determine M. Hence

$$C(M|D) \leq C(m_1, \ldots, m_j|D) \leq \Sigma_{i=1..j} [\tfrac{1}{2} \log n + O(\log\log n)]$$

This implies $f(n) \geq (2n/\log n)[1+O(\log\log n / \log n)]$. QED