Maximal independent set

trivial sequential algorithm

- inherently sequential
- from node point of view: each thinks can join MIS if others stay out
- randomization breaks this symmetry

Randomized idea

- each node joins with some probability
- all neighbors excluded
- many nodes join
- few phases needed

Algorithm:

- all degree 0 nodes join
- node v joins with probability $1/2d(v)$
- if edge (u, v) has both ends marked, unmark lower degree vertex
- put all marked nodes in IS
- delete all neighbors

Intuition: d-regular graph

- vertex vanishes if it or neighbor gets chosen
- mark with probability $1/2d$
- prob (no neighbor marked) is $(1 - 1/2d)^d$, constant
- so const prob. of neighbor of v marked—destroys v
- const fraction of neighbors vanish: $O(\log n)$ iters
- what about unmarking?
- prob(unmarking forced) only constant.
- So just changes constants

Implementing a phase trivial in $O(\log n)$.
Prob chosen for IS, given marked, exceeds 1/2

- suppose w marked. only unmarked if higher degree neighbor marked

- higher degree neighbor marked with prob. $\leq 1/2d(w)$
- only $d(w)$ neighbors
- prob. any marked at most 1/2.
- deduce prob. good vertex killed exceeds $(1 - e^{-1/6})/2$

Good vertices

- good: at least 1/3 neighbors have lower degree
- prob. no neighbor of good marked $\leq (1 - 1/2d(v))^{d(v)/3} \leq e^{-1/6}$.

Good edges

- any edge with a good neighbor
- has const prob. to vanish
- show half edges good
- deduce $O(\log n)$ iterations.

Proof

- Let V_B be bad vertices; we count edges with both ends in V_B.
- direct edges from lower to higher degree d_i is indegree, d_o outdegree
- if v bad, then $d_i(v) \leq d(v)/3$
- deduce
$$\sum_{V_B} d_i(v) \leq \frac{1}{3} \sum_{V_B} d(v) = \frac{1}{3} \sum_{V_B} (d_i(v) + d_o(v))$$
- so $\sum_{V_B} d_i(v) \leq \frac{1}{2} \sum_{V_B} d_o(v)$
- which means indegree can only "catch" half of outdegree; other half must go to good vertices.
- more carefully,
  - $d_o(v) - d_i(v) \geq \frac{1}{3}(d(v)) = \frac{1}{3}(d_o(v) + d_i(v))$.
  - Let V_G, V_B be good, bad vertices
  - degree of bad vertices is
$$\begin{aligned} 2e(V_B, V_B) + e(V_B, V_G) + e(V_G, V_B) &= \sum_{v \in V_B} d_o(v) + d_i(v) \\ &\leq 3 \sum (d_o(v) - d_i(v)) \\ &= 3(e(V_B, V_G) - e(V_G, V_B)) \\ &\leq 3(e(V_B, V_G) + e(V_G, V_B)) \end{aligned}$$

  Deduce $e(V_B, V_B) \leq e(V_B, V_G) + e(V_G, V_B)$. result follows.

Derandomization:

- Analysis focuses on edges,
- so unsurprisingly, pairwise independence sufficient
- not immediately obvious, but again consider d-uniform case
- prob vertex marked $1/2d$
- neighbors $1, \ldots, d$ in increasing degree order
- Let E_i be event that i is marked.
- Let E'_i be E_i but no E_j for $j < i$
- A_i event no neighbor of i chosen
- Then prob eliminate v at least

$$\begin{aligned} \sum \Pr[E'_i \cap A_i] &= \sum \Pr[E'_i] \Pr[A_i \mid E'_i] \\ &\geq \sum \Pr[E'_i] \Pr[A_i] \end{aligned}$$

- Wait: show $\Pr[A_i \mid E'_i] \geq \Pr[A_i]$
  - true if independent
  - measure $\Pr[\neg A_i \mid E'_i] \leq \sum \Pr[E_w \mid E'_i]$
  - measure

$$\begin{aligned} \Pr[E_w \mid E'_i] &= \frac{\Pr[E_w \cap E']}{\Pr[E'_i]} \\ &= \frac{\Pr[E_w \cap \neg E_1 \cap \cdots \mid E_i]}{\Pr[\neg E_1 \cap \cdots \mid E_i]} \\ &\leq \frac{\Pr[E_w \mid E_j]}{1 - \sum \Pr[E_j \mid E_i]} \\ &= \Theta(\Pr[E_i]) \end{aligned}$$

- But expected marked neighbors $1/2$, so by Markov $\Pr[A_i] > 1/2$
- so prob eliminate v exceeds $\sum \Pr[E'_i] = \Pr[\cup E_i]$
- lower bound as $\sum \Pr[E_i] - \sum \Pr[E_i \cap E_j] = 1/2 - d(d-1)/8d^2 > 1/4$
- so $1/2d$ prob. v marked but no neighbor marked, so v chosen
- Generate pairwise independent with $O(\log n)$ bits
- try all polynomial seeds in parallel

- one works
- gives deterministic NC algorithm

with care, $O(m)$ processors and $O(\log n)$ time (randomized)
LFMIS P-complete.

Perfect Matching

We focus on bipartite; book does general case.
Detecting one easy in $\mathcal{NC}$:

- Tutte matrix
- Determinant nonzero iff PM
- Replace vars with values $1, \ldots, 2^m$, same holds
- Matrixu Mul, Determinant in $\mathcal{NC}$
- Wait: big numbers?
- Who cares: poly bits, NC to multiply etc

How about finding one?

- If unique, no problem
- Remove each edge, see if still PM in parallel
- multiplies processors by m
- still NC
- generalizes to polynomial number of matchings

Idea:

- make unique minimum weight perfect matching
- find it

Isolating lemma:

- Family of distinct sets over $x_1, \ldots, x_m$
- assign random weights in $1, \ldots, 2m$
- Pr(unique min-weight set)$\geq 1/2$
- Odd: no dependence on number of sets!
- (of course $< 2^m$)

Proof:

- Fix item x_i
- Y is min-sets containing x_i
- N is min-sets no containing x_i
- true min-sets are either those in Y or in N
- how decide? Value of x_i
- For $x_i = -\infty$, min-sets are Y
- For $x_i = +\infty$, min-sets are N
- As increase from $-\infty$ to ∞, single transition value when both X and Y are min-weight
- If only Y min-weight, then x_i in every min-set
- If only X min-weight, then x_i in no min-set
- If both min-weight, x_i is *ambiguous*
- Suppose no x_i ambiguous. Then min-weight set unique!
- Exactly one value for x_i makes it ambiguous given remainder
- So pr(ambiguous)$1/2m$
- So pr(any ambiguous)$< m/2m = 1/2$

Usage:

- Consider tutte matrix A
- Assign random value 2^{w_i} to x_i, with $w_i \in 1, \ldots, 2m$
- Weight of matching is $2^{\sum w_i}$
- Let W be minimum sum
- Unique w/pr 1/2
- If so, determinant is odd multiple of 2^W
- Try removing edges one at a time
- Edge in PM iff new determinant$/2^W$ is odd.

NC algorithm open.
For exact matching, P algorithm open.

Upcoming

Vempala: “An Eye for Elegance”

- More markov chains
- convex volume estimation
- geometric embeddings
- 11-12:30

Joel Spencer

- 9:30-11
- Probabilstic method
- List of people who took it last time

Spielman advanced complexity
Next year: advanced algorithms.
Bring your research problems