The Probabilistic Method

Idea: to show an object with certain properties exists

- generate a random object
- prove it has properties with nonzero probability
- often, “certain properties” means “good solution to our problem”

Max-Cut:

- Define
- NP-complete
- Approximation algorithms
- factor 2
- “expected performance,” so doesn’t really fit our RP/ZPP framework

Expanders

Existence vs. constriction

- Of course, many probabilistic method constructions yield constructive algorithms
- In maxcut, just try till succeed
- Other times, are only existential proofs, or very bad algorithms
- But motivate search for good algorithm

Definition: (n, d, α, c) OR-concentrator

- bipartite $2n$ vertices
- degree at most d in L
- expansion c on sets $< \alpha n$.

Applications:

- switching/routing
- ECCs

claim: $(n, 18, 1/3, 2)$-concentrator

- Construct by sampling d random neighbors with replacement
  - E_s: Specific size s set has $< cs$ neighbors.

- fix S of size s. T of size $< cs$.
- prob. S goes to T at most $(cs/n)^{ds}$
- $\binom{n}{cs}$ sets T
- $\binom{n}{s}$ sets S
-

$$\begin{aligned}
\Pr[] &\leq \binom{n}{s}\binom{n}{cs}(cs/n)^{ds} \\
&\leq (en/s)^s (en/cs)^{cs} (cs/n)^{ds} \\
&= [(s/n)^{d-c-1} e^{c+1} c^{d-c}]^s \\
&\leq [(1/3)^{d-c-1} e^{c+1} c^{d-c}]^s \\
&\leq [(c/3)^d (3e)^{c+1}]^s
\end{aligned}$$

- Take $c = 2, d = 18$, get $[(2/3)^{18}(3e)^3]^{<} 2^{-s}$
- sum over s, get < 1

Existence proof

- No known construction this good.
- NP-hard to verify
- but some constructions almost this good

Wiring

Sometimes, it's hard to get hands on a good probability distribution.

- Problem formulation
  - $\sqrt{n} \times \sqrt{n}$ *gate array*
  - Manhattan wiring
  - boundaries between gates
  - fixed width boundary means limit on number of crossing wires
  - optimization vs. feasibility: minimize max crossing number
  - focus on single-bend wiring. two choices for route.
  - Generalizes if you know about max-flow
- Linear Programs, integer linear programs
  - Black box
  - Good to know, since great solvers exist in practice

- Solution techniques in other courses

- IP formulation
  - x_{i0} means x_i starts horizontal, x_{i1} vertical
  - $T_{b0} = \{i \mid \text{net } i \text{ through } b \text{ if } x_{i0}\}$
  - T_{b1}
  - IP

$$\begin{aligned} \min \quad & w \\ x_{i0} + x_{i1} &= 1 \\ \sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} &\leq w \end{aligned}$$

- Solution $\hat{x}_{i0}$, $\hat{x}_{i1}$, value $\hat{w}$.

- rounding is Poisson vars, mean $\hat{w}$.

- $\Pr[\geq (1+\delta)\hat{w}] \leq e^{-\delta^2 \hat{w}/4}$

- need $2n$ boundaries, so aim for prob. bound $1/2n^2$.

- solve, $\delta = \sqrt{(4 \ln 2n^2)/\hat{w}}$.

- So absolute error $\sqrt{8\hat{w} \ln n}$
  - Good ($o(1)$-error) if $\hat{w} \gg 8 \ln n$
  - Bad ($O(\ln n)$ error) is $\hat{w} = 2$
  - General rule: randomized rounding good if target logarithmic, not if constant

MAX SAT

Define.

- literals

- clauses

- NP-complete

random set

- achieve $1 - 2^{-k}$

- very nice for large k, but only $1/2$ for $k = 1$

LP

$$\max \quad \sum z_j$$

$$\sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_1) \geq z_j$$

Analysis

- $\beta_k = 1 - (1 - 1/k)^k$. values $1, 3/4, .704, \ldots$
- Lemma: k-literal clause sat w/pr at least $\beta_k \hat{z}_j$.
- proof:
  - assume all positive literals.
  - prob $1 - \prod(1 - y_i)$
  -
  - maximize when all $y_i = \hat{z}_j/k$.
  - Show $1 - (1 - \hat{z}/k)^k \geq \beta_k \hat{z}_k$.
  - check at $z = 0, 1$
- Result: $(1 - 1/e)$ approximation (convergence of $(1 - 1/k)^k$)
- much better for small k: i.e. 1-approx for $k = 1$

LP good for small clauses, random for large.

- Better: try both methods.
- n_1, n_2 number in both methods
- Show $(n_1 + n_2)/2 \geq (3/4) \sum \hat{z}_j$
- $n_1 \geq \sum_{C_j \in S^k} (1 - 2^{-k}) \hat{z}_j$
- $n_2 \geq \sum \beta_k \hat{z}_j$
- $n_1 + n_2 \geq \sum (1 - 2^{-k} + \beta_k) \hat{z}_j \geq \sum \frac{3}{2} \hat{z}_j$