Introduction to Quantum Information Processing

Lecture 4

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Overview

- Von Neumann measurements
- General measurements
- Traces and density matrices and partial traces

"Von Neumann measurement in the computational basis"

- Suppose we have a universal set of quantum gates, and the ability to measure each qubit in the basis $\{|0\rangle, |1\rangle\}$
- If we measure $|\Phi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle)$ we get $|b\rangle$ with probability $|\alpha_b|^2$

In section 2.2.5, this is described as follows

- We have the projection operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ satisfying $P_0 + P_1 = I$
- We consider the projection operator or "observable" $M = 0P_0 + 1P_1 = P_1$
- Note that 0 and 1 are the eigenvalues
- When we measure this observable M, the probability of getting the eigenvalue b is $\Pr(b) = \langle \Phi | P_b | \Phi \rangle = |\alpha_b|^2$ and we are in that case left with the state $\frac{P_b|\Phi\rangle}{\sqrt{p(b)}} = \frac{\alpha_b}{|\alpha_b|}|b\rangle \approx |b\rangle$

"Expected value" of an observable

If we associate with outcome $|b\rangle$ the eigenvalue b then the expected outcome is

$$\sum_b b \Pr(b)$$

$$= \sum_b b \langle \Phi | P_b | \Phi \rangle = \langle \Phi | \left(\sum_b b P_b \right) | \Phi \rangle$$

$$= Tr\left[\langle \Phi | \left(\sum_b b P_b \right) | \Phi \rangle \right] = Tr\left[M |\Phi\rangle\langle \Phi| \right]$$

"Von Neumann measurement in the computational basis"

- Suppose we have a universal set of quantum gates, and the ability to measure each qubit in the basis $\{|0\rangle, |1\rangle\}$
- Say we have the state $\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$
- If we measure all n qubits, then we obtain $|x\rangle$ with probability $|\alpha_x|^2$
- Notice that this means that probability of measuring a $|0\rangle$ in the first qubit equals $\sum_{x \in 0\{0,1\}^{n-1}} |\alpha_x|^2$

Partial measurements

- If we only measure the first qubit and leave the rest alone, then we still get $|0\rangle$ with probability $p_0 = \sum_{x \in 0\{0,1\}^{n-1}} |\alpha_x|^2$
- The remaining n-1 qubits are then in the renormalized state $$\sum_{x \in 0\{0,1\}^{n-1}} \frac{\alpha_x}{\sqrt{p_0}} |x\rangle$$
- (This is similar to Bayes Theorem)

In section 2.2.5

- This partial measurement corresponds to measuring the observable

$$M = 0|0\rangle\langle 0| \otimes I^{n-1} + 1|1\rangle\langle 1| \otimes I^{n-1}$$

Von Neumann Measurements

- A Von Neumann measurement is a type of projective measurement. Given an orthonormal basis $\{|\psi_k\rangle\}$, if we perform a Von Neumann measurement with respect to $\{|\psi_k\rangle\}$ of the state $|\Phi\rangle = \sum \alpha_k |\psi_k\rangle$ then we measure $|\psi_k\rangle$ with probability

$$|\alpha_k|^2 = |\langle \psi_k | \Phi \rangle|^2 = \langle \psi_k | \Phi \rangle \langle \Phi | \psi_k \rangle$$
$$= \mathrm{Tr}\left(\langle \psi_k | \Phi \rangle \langle \Phi | \psi_k \rangle\right) = \mathrm{Tr}\left(|\psi_k\rangle\langle \psi_k || \Phi \rangle \langle \Phi |\right)$$

Von Neumann Measurements

- E.x. Consider Von Neumann measurement of the state $|\Phi\rangle = (\alpha|0\rangle + \beta|1\rangle)$ with respect to the orthonormal basis $\left\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}$
- Note that

$$|\Phi\rangle = \frac{\alpha+\beta}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + \frac{\alpha-\beta}{\sqrt{2}}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

- We therefore get $\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$ with probability $\frac{|\alpha+\beta|^2}{2}$

Von Neumann Measurements

- Note that $\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)|\Phi\rangle = \frac{\alpha+\beta}{\sqrt{2}}$

$$\langle\Phi|\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{\alpha^*+\beta^*}{\sqrt{2}}$$

$$\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)|\Phi\rangle\langle\Phi|\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

$$= Tr\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)|\Phi\rangle\langle\Phi|\right) = \frac{|\alpha+\beta|^2}{2}$$

How do we implement Von Neumann measurements?

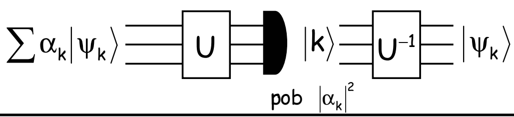
- If we have access to a universal set of gates and bit-wise measurements in the computational basis, we can implement Von Neumann measurements with respect to an arbitrary orthonormal basis $\{|\psi_k\rangle\}$ as follows.

How do we implement Von Neumann measurements?

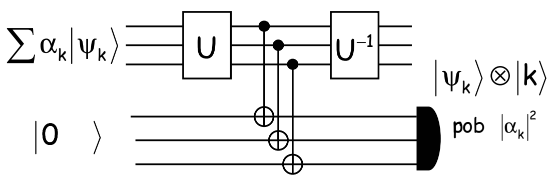
- Construct a quantum network that implements the unitary transformation

$$U|\psi_k\rangle = |k\rangle$$

- Then "conjugate" the measurement operation with the operation U



Another approach



$$\sum \alpha_k |\psi_k\rangle \otimes |0\quad\rangle \rightarrow \sum \alpha_k |k\rangle \otimes |0\quad\rangle$$
$$\rightarrow \sum \alpha_k |k\rangle \otimes |k\rangle \rightarrow \sum \alpha_k |\psi_k\rangle \otimes |k\rangle$$

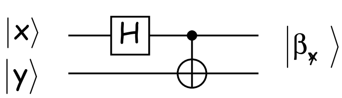
Ex. Bell basis change

- Consider the orthonormal basis consisting of the "Bell" states

$|\beta_0\ \rangle = |0\ \ \rangle + |1\ \ \rangle \qquad |\beta_0\ \rangle = |0\ \ \rangle + |0\ \ \rangle$

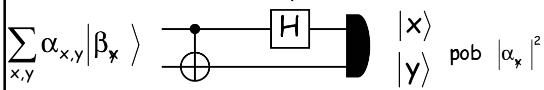
$|\beta_0\ \rangle = |0\ \ \rangle - |1\ \ \rangle \qquad |\beta_1\ \rangle = |0\ \ \rangle - |0\ \ \rangle$

- Note that

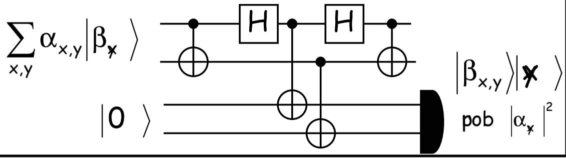


Bell measurement

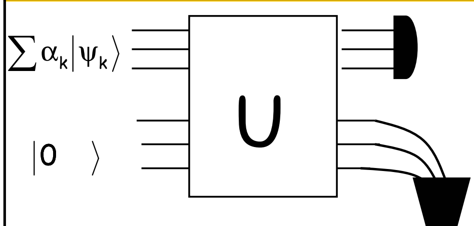
- We can "destructively" measure



- Or non-destructively project



Most general measurement



Trace of a matrix

The trace of a matrix is the sum of its diagonal elements

e.g. $$Tr\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

Some properties:
$$Tr[xA + yB] = xTr[A] + yTr[B]$$
$$Tr[AB] = Tr[BA]$$
$$Tr[ABC] = Tr[CAB]$$
$$Tr[UAU^{\dagger}] = Tr[A]$$

Orthonormal basis { $|\varphi_i\rangle$ } $\quad Tr[A] = \sum_i \langle \varphi_i | A | \varphi_i \rangle$

Density Matrices

$$|\varphi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

Notice that $\alpha_0 = \langle 0|\phi\rangle$, and $\alpha_1 = \langle 1|\phi\rangle$.

So the probability of getting 0 when measuring $|\phi\rangle$ is:

$$p(0) = |\alpha_0|^2 = |\langle 0|\phi\rangle|^2$$

$$= \langle 0|\varphi\rangle(\langle 0|\varphi\rangle)^* = \langle 0|\varphi\rangle\langle\varphi|0\rangle$$

$$= \langle 0||\varphi\rangle\langle\varphi||0\rangle = Tr(\langle 0||\varphi\rangle\langle\varphi||0\rangle)$$

$$= Tr(|0\rangle\langle 0||\varphi\rangle\langle\varphi|) = Tr(|0\rangle\langle 0|\rho)$$

where $\rho = |\phi\rangle\langle\phi|$ is called the *density matrix* for the state $|\phi\rangle$

Mixture of pure states

A state described by a state vector $|\phi\rangle$ is called a *pure state*.

What if we have a qubit which is known to be in the pure state $|\phi_1\rangle$ with probability p_1, and in $|\phi_2\rangle$ with probability p_2 ?
More generally, consider probabilistic mixtures of pure states (called *mixed states*):

$$\varphi = \{(|\varphi_1\rangle, p_1), (|\varphi_2\rangle, p_2), \ldots\}$$

Density matrix of a mixed state

…then the probability of measuring 0 is given by conditional probability:

$$p(0) = \sum_i p_i \cdot (\text{prob. of measuring 0 given pure state } |\phi_i\rangle)$$

$$= \sum_i p_i \cdot Tr(|0\rangle\langle 0||\varphi_i\rangle\langle\varphi_i|)$$

$$= Tr\sum_i p_i|0\rangle\langle 0||\varphi_i\rangle\langle\varphi_i|$$

$$= Tr(|0\rangle\langle 0|\rho)$$

where $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$ is the *density matrix* for the mixed state ϕ

Density matrices contain all the useful information about an arbitrary quantum state.

Density Matrix

If we apply the unitary operation U to $|\psi\rangle$ the resulting state is $U|\psi\rangle$ with density matrix

$$U|\psi\rangle(U|\psi\rangle)^\dagger = U|\psi\rangle\langle\psi|U^\dagger$$

Density Matrix

If we apply the unitary operation U to $\{(q_k, |\psi_k\rangle)\}$ the resulting state is $\{(q_k, U|\psi_k\rangle)\}$ with density matrix

$$\sum_k q_k U|\psi_k\rangle\langle\psi_k|U^\dagger$$
$$= U\left(\sum_k q_k|\psi_k\rangle\langle\psi_k|\right)U^\dagger$$
$$= U\rho U^\dagger$$

Density Matrix

If we perform a Von Neumann measurement of the state $\rho = |\psi\rangle\langle\psi|$ wrt a basis containing $|\phi\rangle$, the probability of obtaining $|\phi\rangle$ is

$$|\langle\psi|\phi\rangle|^2 = Tr\left(\rho|\phi\rangle\langle\phi|\right)$$

Density Matrix

If we perform a Von Neumann measurement of the state $\{(q_k, |\psi_k\rangle)\}$ wrt a basis containing $|\phi\rangle$ the probability of obtaining $|\phi\rangle$ is

$$\sum_k q_k |\langle\psi_k|\varphi\rangle|^2 = \sum_k q_k Tr(|\psi_k\rangle\langle\psi_k\|\varphi\rangle\langle\varphi|)$$

$$= Tr\left(\sum_k q_k |\psi_k\rangle\langle\psi_k\|\varphi\rangle\langle\varphi|\right)$$

$$= Tr(\rho|\varphi\rangle\langle\varphi|)$$

Density Matrix

In other words, the density matrix contains all the information necessary to compute the probability of any outcome in any future measurement.

Spectral decomposition

- Often it is convenient to rewrite the density matrix as a mixture of its eigenvectors
- Recall that eigenvectors with distinct eigenvalues are orthogonal; for the subspace of eigenvectors with a common eigenvalue ("degeneracies"), we can select an orthonormal basis

Spectral decomposition

- In other words, we can always "diagonalize" a density matrix so that it is written as

$$\rho = \sum_k p_k |\varphi_k\rangle\langle\varphi_k|$$

where $|\varphi_k\rangle$ is an eigenvector with eigenvalue p_k and $\{|\varphi_k\rangle\}$ forms an orthonormal basis

Partial Trace

- How can we compute probabilities for a partial system?
- E.g. $\sum_{x,y} \alpha_{xy} |x\rangle|y\rangle$

$$= \sum_y \left(\sum_x \alpha_{xy} |x\rangle \right) |y\rangle$$

$$= \sum_y \sqrt{p_y} \left(\sum_x \frac{\alpha_{xy}}{\sqrt{p_y}} |x\rangle \right) |y\rangle$$

Partial Trace

- If the 2nd system is taken away and never again (directly or indirectly) interacts with the 1st system, then we can treat the first system as the following mixture
- E.g. $\sum_y \sqrt{p_y} \left(\sum_x \frac{\alpha_{xy}}{\sqrt{p_y}} |x\rangle \right) |y\rangle \approx \rho$

$$\xrightarrow{Trace_2} \left\{ \left(p_y, \sum_x \frac{\alpha_{xy}}{\sqrt{p_y}} |x\rangle \right) \right\} \approx \rho_2 = Tr_2 \rho$$

Partial Trace

$$\sum_y \sqrt{p_y}\left(\sum_x \frac{\alpha_{xy}}{\sqrt{p_y}}|x\rangle\right)|y\rangle \approx \rho$$

$$\xrightarrow{Trace_2}\left\{\left(p_y, \sum_x \frac{\alpha_{xy}}{\sqrt{p_y}}|x\rangle\right)\right\} \approx \rho_2 = Tr_2\rho$$

$$Tr_2\rho = \sum_y p_y |\Phi_y\rangle\langle\Phi_y| \qquad |\Phi_y\rangle = \sum_x \frac{\alpha_{xy}}{\sqrt{p_y}}|x\rangle$$

Why?

- the probability of measuring e.g. $|w\rangle$ in the first register depends only on $Tr_2\rho$

$$\sum_y |\alpha_{wy}|^2 = \sum_y p_y \left|\frac{\alpha_{wy}}{\sqrt{p_y}}\right|^2$$

$$= \sum_y p_y Tr(|w\rangle\langle w||\Phi_y\rangle\langle\Phi_y|)$$

$$= Tr\left(|w\rangle\langle w|\left(\sum_y p_y|\Phi_y\rangle\langle\Phi_y|\right)\right)$$

$$= Tr(|w\rangle\langle w|Tr_2\rho)$$

Partial Trace

- Notice that it doesn't matter in which orthonormal basis we "trace out" the 2nd system, e.g.

$$\alpha|00\rangle + \beta|11\rangle \xrightarrow{Tr_2} |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

- In a different basis

$$\alpha|00\rangle + \beta|11\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$+\frac{1}{\sqrt{2}}(\alpha|0\rangle - \beta|1\rangle)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

Partial Trace

$$\frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)$$
$$+\frac{1}{\sqrt{2}}(\alpha|0\rangle-\beta|1\rangle)\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right)$$
$$\xrightarrow{Tr_2}\frac{1}{2}(\alpha|0\rangle+\beta|1\rangle)(\alpha^*\langle 0|+\beta^*\langle 1|)$$
$$+\frac{1}{2}(\alpha|0\rangle-\beta|1\rangle)(\alpha^*\langle 0|-\beta^*\langle 1|)$$
$$=|\alpha|^2|0\rangle\langle 0|+|\beta|^2|1\rangle\langle 1|$$

Distant transformations don't change the local density matrix

- Notice that the previous observation implies that a unitary transformation on the system that is traced out does not affect the result of the partial trace
- I.e.

$$\sum_y \sqrt{p_y}|\Phi_y\rangle U|y\rangle \approx (I\otimes U)\rho$$
$$\xrightarrow{Trace_2}\{(p_{y,}|\Phi_y\rangle)\}\approx \rho_2 = Tr_2\rho$$

Distant transformations don't change the local density matrix

- In fact, any legal quantum transformation on the traced out system, including measurement (without communicating back the answer) does not affect the partial trace
- I.e.

$$\{(p_y,|\Phi_y\rangle|y\rangle)\}$$
$$\xrightarrow{Trace_2}\{(p_{y,}|\Phi_y\rangle)\}\approx \rho_2 = Tr_2\rho$$

Why??

- Operations on the 2nd system should not affect the statistics of any outcomes of measurements on the first system
- Otherwise a party in control of the 2nd system could instantaneously communicate information to a party controlling the 1st system.

Principle of implicit measurement

- If some qubits in a computation are never used again, you can assume (if you like) that they have been measured (and the result ignored)
- The "reduced density matrix" of the remaining qubits is the same

Partial Trace

- This is a linear map that takes bipartite states to single system states.
- We can also trace out the first system
- We can compute the partial trace directly from the density matrix description

$$Tr_2\left(|i\rangle\langle k| \otimes |j\rangle\langle l|\right) = |i\rangle\langle k| \otimes Tr\left(|j\rangle\langle l|\right)$$
$$= |i\rangle\langle k| \otimes \langle l|j\rangle = \langle l|j\rangle |i\rangle\langle k|$$

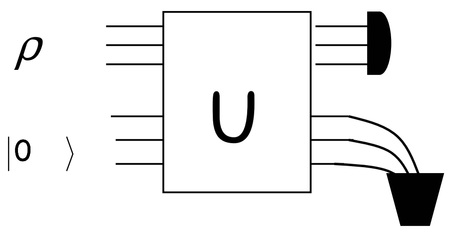
Partial Trace using matrices

- Tracing out the 2nd system

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_2} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00} + a_{11} & a_{02} + a_{13} \\ a_{20} + a_{31} & a_{22} + a_{33} \end{bmatrix}$$

Most general measurement



$$\rho \mapsto Tr_2\left(\rho \otimes |0\rangle\langle 0|\right)$$