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Introduction to Quantum Information Processing

Lecture 9

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Overview

- Dirac Notation comment
- Partial Trace and Schmidt Decomposition
- The Bloch Ball, one-qubit gates, and controlled-U

Dirac notation quirk

- When taking tensor products of subsystems, we can clarify which vectors correspond to which subsystem
- E.g. $|i\rangle_1 |j\rangle_2$ means system 1 is in state $|i\rangle$ and system 2 is in state $|j\rangle$
- When computing the conjugate transpose, following standard matrix convention we would write

$$\left(|i\rangle_1 |j\rangle_2\right)^t = \langle i|_1 \langle j|_2$$

Dirac notation quirk

- However, it is more common in physics to write (often without the subscripts)

$$\left(|i\rangle_1 |j\rangle_2\right)^{t} = \langle j|_2 \langle i|_1$$

- This way we can e.g. compute an inner product in the following way

$$\left(|i\rangle_1 |j\rangle_2\right)^{t} |k\rangle_1 |l\rangle_2 = \langle j|_2 \langle i|_1 |k\rangle_1 |l\rangle_2$$
$$= \langle j|_2 \langle i|k\rangle |l\rangle_2 = \langle i|k\rangle \langle j|_2 |l\rangle_2$$
$$= \langle i|k\rangle \langle j|l\rangle$$

Partial Trace

- Partial trace is the linear extension of the following map:

$$Tr_2(A \otimes B) = A Tr(B)$$

- In Dirac notation:

$$Tr_2\left(|i\rangle\langle k| \otimes |j\rangle\langle l|\right) = |i\rangle\langle k| \otimes Tr\left(|j\rangle\langle l|\right)$$
$$= |i\rangle\langle k| \otimes \langle l|j\rangle = \langle l|j\rangle |i\rangle\langle k|$$

- Note $|i\rangle\langle k| \otimes |j\rangle\langle l| = |i\rangle|j\rangle\left(|k\rangle|l\rangle\right)^{t}$
- Can see this by recalling

$$\left(AB^{t}\right) \otimes \left(CD^{t}\right) = (A \otimes C)\left(B^{t} \otimes D^{t}\right) = (A \otimes C)(B \otimes D)^{t}$$

Partial Trace using matrices

- Tracing out the 2nd system

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_2} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} a_{00} + a_{11} & a_{02} + a_{13} \\ a_{20} + a_{31} & a_{22} + a_{33} \end{bmatrix}$$

Partial Trace using matrices

- Tracing out the 1st system

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_1} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{02} \\ a_{20} & a_{22} \end{bmatrix} & Tr\begin{bmatrix} a_{01} & a_{03} \\ a_{21} & a_{23} \end{bmatrix} \\ Tr\begin{bmatrix} a_{10} & a_{12} \\ a_{30} & a_{32} \end{bmatrix} & Tr\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00} + a_{22} & a_{01} + a_{23} \\ a_{10} + a_{32} & a_{11} + a_{33} \end{bmatrix}$$

Schmidt decomposition (section 2.5)

- Theorem: If $|\psi\rangle$ is a pure state of a composite system AB, then there exists an orthonormal basis $\{|\Phi_i^A\rangle\}$ for system A and $\{|\Phi_i^B\rangle\}$ for system B, and non-negative reals $\{p_i\}$, so that

$$|\psi\rangle = \sum_i \sqrt{p_i}\,|\Phi_i^A\rangle|\Phi_i^B\rangle$$

$\lambda_i = \sqrt{p_i}$ "Schmidt coefficients"

Schmidt decomposition trivial example

- E.g. $|\psi\rangle = |11\rangle$

$$\{|\Phi_0^A\rangle = |1\rangle, |\Phi_1^A\rangle = |0\rangle\}$$

$$\{|\Phi_0^B\rangle = |1\rangle, |\Phi_1^B\rangle = |0\rangle\}$$

$$\{p_0 = 1, p_1 = 0\}$$

Schmidt decomposition almost trivial example

- E.g. $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

$$= 1\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\left\{|\Phi_0^A\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{1}}|1\rangle, |\Phi_1^A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right\}$$

$$\left\{|\Phi_0^B\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle, |\Phi_1^B\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right\}$$

$$\{p_0 = 1, p_1 = 0\}$$

Schmidt decomposition example

- E.g.

$$|\psi\rangle = \frac{1+\sqrt{6}}{2\sqrt{6}}|00\rangle + \frac{1-\sqrt{6}}{2\sqrt{6}}|01\rangle + \frac{\sqrt{2}-\sqrt{3}}{2\sqrt{6}}|10\rangle + \frac{\sqrt{2}+\sqrt{3}}{2\sqrt{6}}|11\rangle$$

$$\left\{|\Phi_0^A\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle, |\Phi_1^A\rangle = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle\right\}$$

$$\left\{|\Phi_0^B\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, |\Phi_1^B\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right\}$$

$$\left\{p_0 = \frac{1}{4}, p_1 = \frac{3}{4}\right\} \qquad |\psi\rangle = \sqrt{p_0}|\Phi_0^A\rangle|\Phi_0^B\rangle + \sqrt{p_1}|\Phi_1^A\rangle|\Phi_1^B\rangle$$

Schmidt decomposition application

- It is very easy to compute the reduced density matrices given the Schmidt decomposition

$$|\psi\rangle = \sum_i \sqrt{p_i}|\Phi_i^A\rangle|\Phi_i^B\rangle$$

$$Tr_2|\psi\rangle\langle\psi| = \sum_i p_i|\Phi_i^A\rangle\langle\Phi_i^A|$$

$$Tr_1|\psi\rangle\langle\psi| = \sum_i p_i|\Phi_i^B\rangle\langle\Phi_i^B|$$

observations

- Notice that the spectrum (i.e. set of eigenvalues) of both reduced density matrices are the same

$$Tr_2|\psi\rangle\langle\psi| = \sum_i p_i |\Phi_i^A\rangle\langle\Phi_i^A|$$
$$Tr_1|\psi\rangle\langle\psi| = \sum_i p_i |\Phi_i^B\rangle\langle\Phi_i^B|$$

How do we compute the Schmidt decomposition?

- Nielsen and Chuang recommend the Singular Value Decomposition; very elegant
- Alternatively, compute the partial traces, and diagonalize them in order to find the correct bases for each subsystem
- Or guess.

Other observations

- Read exercises 2.80, 2.81, 2.82 for other very important facts that can be proved easily using the Schmidt decomposition (we will discuss these more later when relevant).

Bloch Sphere

- These 4 matrices form a basis for the 2x2 density matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- So every density matrix can be written as

$$\rho = \frac{1}{2}\left(I + a_x X + a_y Y + a_z Z\right)$$

Bloch Ball

- We associate with every 1-qubit state $\rho = \frac{1}{2}(I + a_x X + a_y Y + a_z Z)$ the vector (a_x, a_y, a_z)
- If $\rho = |\Psi\rangle\langle\Psi|$ for a pure state $|\Psi\rangle = e^{i\alpha}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right)$

Then the corresponding vector is

$(a_x, a_y, a_z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

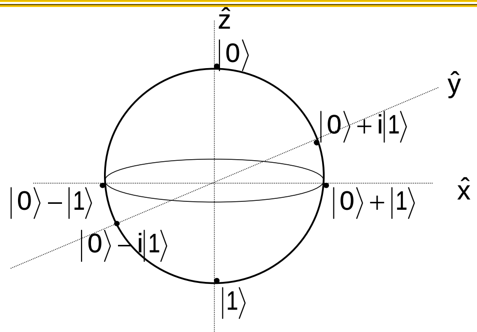
Bloch Sphere

- Notice that the vectors $(a_x, a_y, a_z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

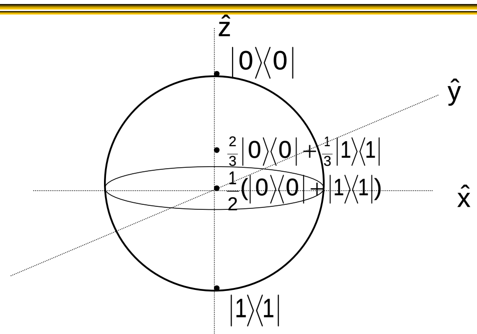
satisfy $|a_x|^2 + |a_y|^2 + |a_z|^2 = 1$

i.e. pure states lie on the surface of the Bloch Ball. By convexity, mixed states lie within the Bloch Ball.

Bloch Sphere



Mixed States



Bloch Ball

- Rotations about the $\hat{x}$ axis are denoted

$$R_x(\theta) = e^{-i\theta X/2} = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right) X = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

- Similar definitions for rotations about the $\hat{y}$ and $\hat{z}$ axes (section 4.2)

Bloch Ball

- We can define a rotation about any axis $\hat{n} = (n_x, n_y, n_z)$

$$R_{\hat{n}}(\theta) = e^{-i\theta\hat{n}\cdot(X,Y,Z)/2}$$
$$= \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z)$$

Bloch Ball

- Alternatively, we can describe these rotations as $R_{\alpha,\varphi}(\theta)$ where

$$|\Psi\rangle = \cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\alpha}{2}\right)|1\rangle$$
$$|\Psi^{\perp}\rangle = \sin\left(\frac{\alpha}{2}\right)|0\rangle - e^{i\varphi}\cos\left(\frac{\alpha}{2}\right)|1\rangle$$
$$R_{\alpha,\varphi}(\theta)|\Psi\rangle = |\Psi\rangle$$
$$R_{\alpha,\varphi}(\theta)|\Psi^{\perp}\rangle = e^{i\theta}|\Psi^{\perp}\rangle$$

Arbitrary 1-qubit operations

- Theorem (Exercise 4.8): Any 1-qubit operation can be written in the form

$$U = e^{i\alpha}R_{\hat{n}}(\theta)$$

for some axis $\hat{n}$ and angle θ (the "global phase" is not important… yet).

Arbitrary 1-qubit operations

- Theorem 4.1: Any 1-qubit operation can be written in the form

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

For some real numbers $\alpha, \beta, \gamma, \delta$

Arbitrary 1-qubit operations

- Corollary 4.2: Any 1-qubit operation can be written in the form

$$U = e^{i\alpha} AXBXC$$

where A, B, C are unitary operators satisfying $ABC = I$

(this comes in handy when construction the controlled-U)

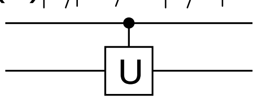
The controlled-U

- The controlled-U or C(U) corresponds to the operation

$$C(U)|0\rangle|\Psi\rangle = |0\rangle|\Psi\rangle$$

$$C(U)|1\rangle|\Psi\rangle = |1\rangle U|\Psi\rangle$$

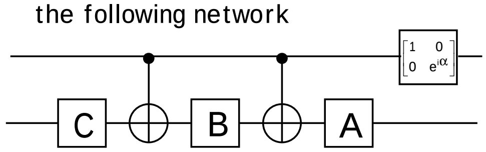
denoted



(note that how we define the "global" phase of U significantly affects the C(U))

The controlled-U

- We can realize the controlled-U with the following network



- So controlled-NOT plus all 1-qubit gates allow us to implement any controlled-U gate

The controlled-U

- It helps to observe that