Introduction to Quantum Information Processing

Lecture 5

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Overview of Lecture 5

- Review of some introductory material: quantum states, operations, and simple quantum circuits
- Communication tasks:
  - one qubit conveys at most one bit
  - superdense coding
  - teleportation

review of introductory material

Classical & quantum states

Probabilistic states:

$$\forall x,\ p_x \geq 0$$

$$\sum_x p_x = 1$$

$$\begin{bmatrix} p_{000} \\ p_{001} \\ p_{010} \\ p_{011} \\ p_{100} \\ p_{101} \\ p_{110} \\ p_{111} \end{bmatrix}$$

Quantum states:

$$\forall x,\ \alpha_x \in \mathbb{C}$$

$$\sum_x |\alpha_x|^2 = 1$$

$$\begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{101} \\ \alpha_{110} \\ \alpha_{111} \end{bmatrix}$$

Dirac notation: $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$

Dirac bra-c-ket notation

Ket: $|\psi\rangle$ always denotes a column vector, e.g. $\begin{bmatrix} \alpha_{000} \\ \alpha_{001} \\ \vdots \\ \alpha_{111} \end{bmatrix}$

Bra: $\langle\psi|$ always denotes a row vector that is the conjugate transpose of $|\psi\rangle$, e.g. $[\ \alpha^*_{000} \quad \alpha^*_{001} \quad \ldots \quad \alpha^*_{111}\ ]$

Bracket: $\langle\varphi|\psi\rangle$ denotes $\langle\varphi|\cdot|\psi\rangle$, the inner product of $|\varphi\rangle$ and $|\psi\rangle$

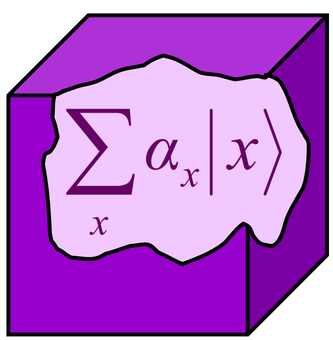
Some basic quantum operations

Initialize: set a qubit to state $|0\rangle$ or $|1\rangle$

Unitary operations: $\sum_x \alpha_x |x\rangle \quad \mapsto \quad U\left(\sum_x \alpha_x |x\rangle\right)$

$(U^\dagger U = I)$

Measurements:



$$\mapsto \begin{cases} 000 & \text{with prob } |\alpha_{000}|^2 \\ 001 & \text{with prob } |\alpha_{001}|^2 \\ \vdots & \vdots \quad\quad \vdots \\ 111 & \text{with prob } |\alpha_{111}|^2 \end{cases}$$

… and the quantum state collapses

Examples: one-qubit operations

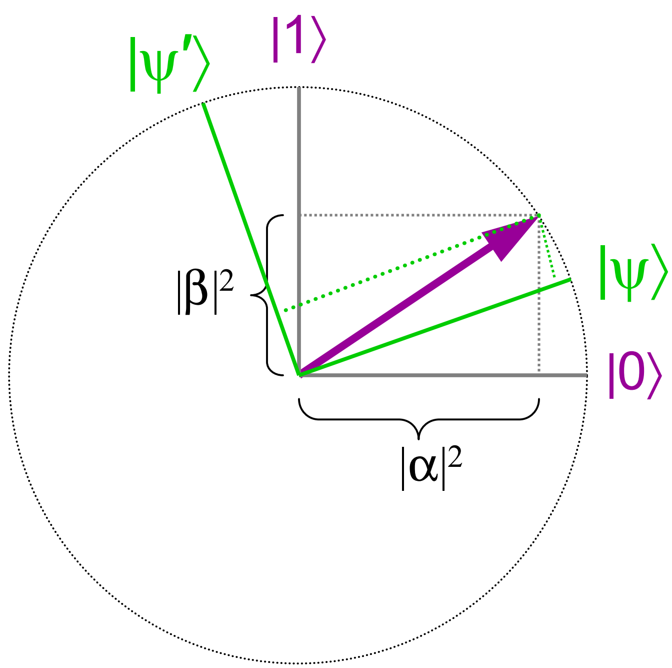
Rotation: $\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}$

NOT (bit flip): $\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Phase flip: $\sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Hadamard: $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Example: measuring a qubit

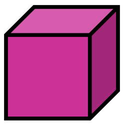


With respect to the "computational basis" $\{|0\rangle, |1\rangle\}$

There exist ***other*** quantum operations, but they can all be "simulated" by the aforementioned types

Example: measurement with respect to a different orthonormal basis $\{ |\psi\rangle, |\psi'\rangle \}$

Distinguishing between two states

Let  be in state $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ or $|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

Question 1: can we distinguish between the two cases?

Distinguishing procedure:

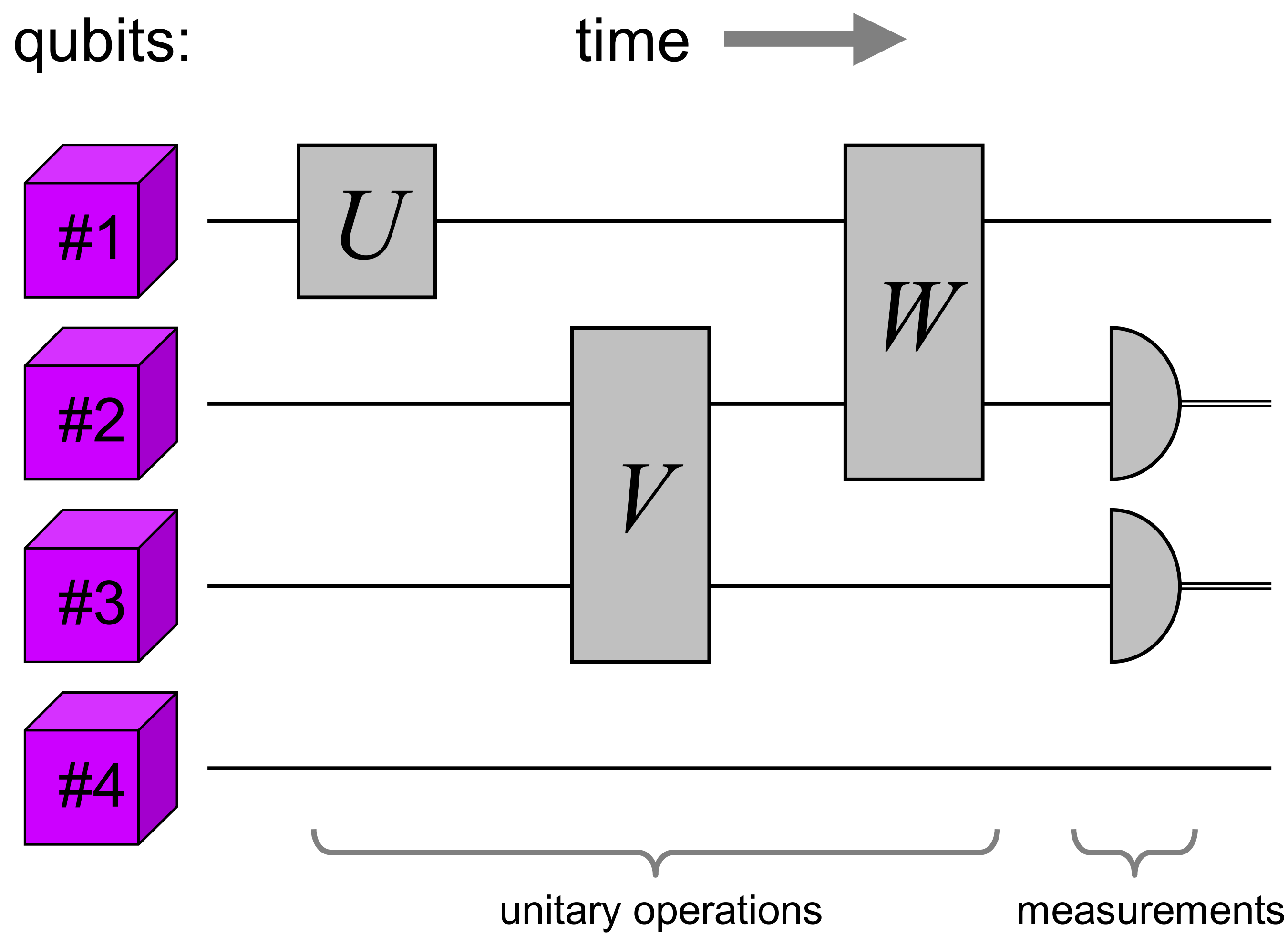
1. apply H
2. measure

This works because $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$

Question 2: can we distinguish between $|0\rangle$ and $|+\rangle$?

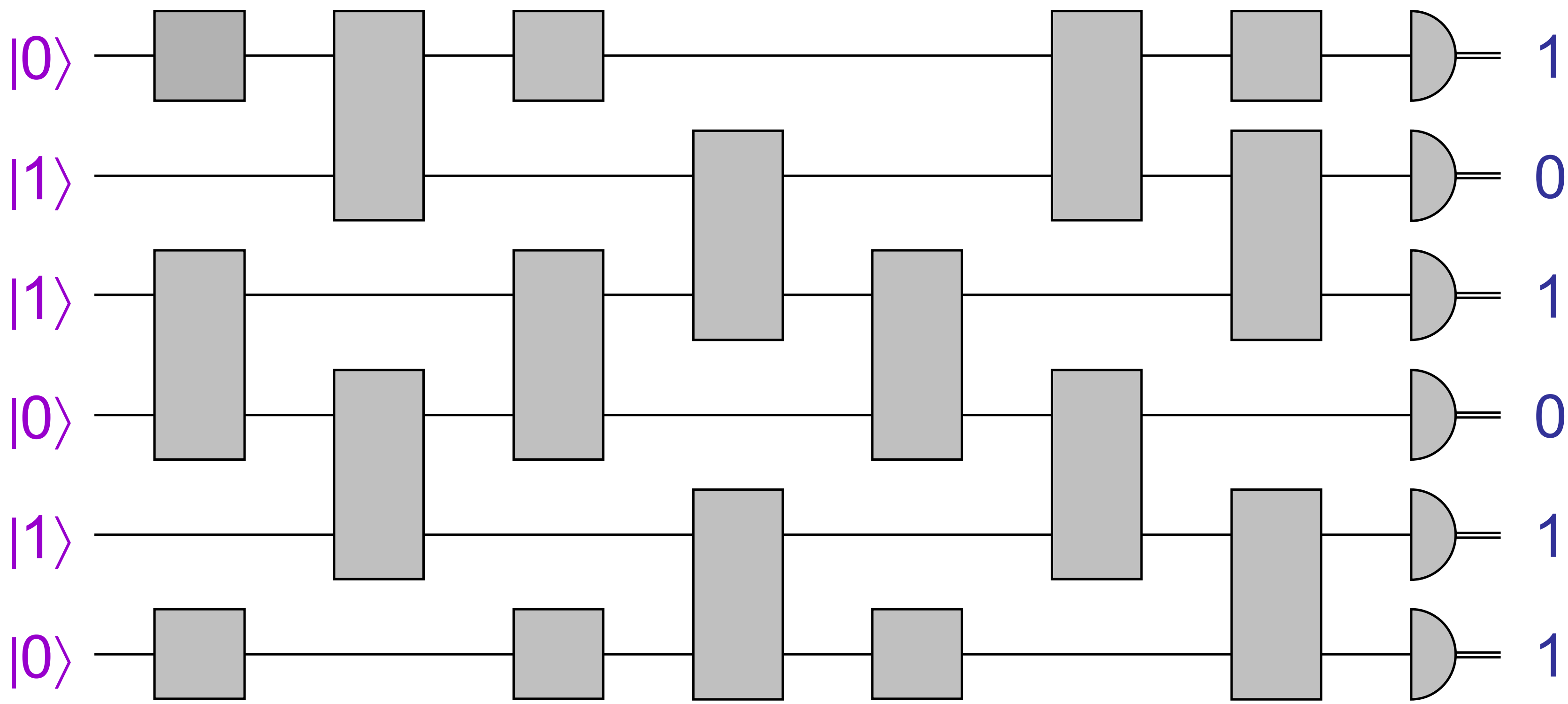
It turns out that, since they're not orthogonal, they ***cannot*** be ***perfectly*** distinguished …

Structure among subsystems

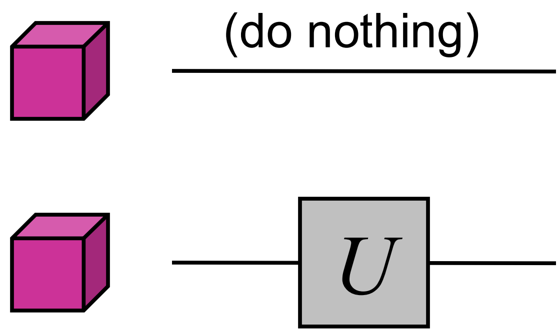


Quantum computations

Quantum circuits:



Examples of two-qubit systems



$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$

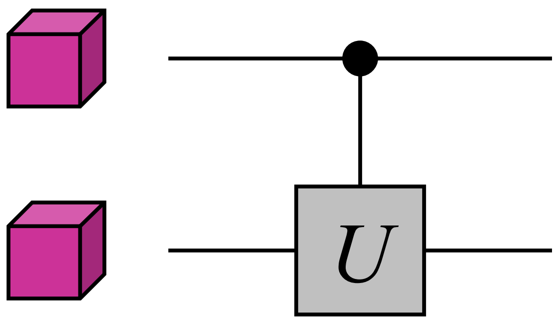
Maps basis states as:

$$|0\rangle|0\rangle \rightarrow |0\rangle U|0\rangle$$
$$|0\rangle|1\rangle \rightarrow |0\rangle U|1\rangle$$
$$|1\rangle|0\rangle \rightarrow |1\rangle U|0\rangle$$
$$|1\rangle|1\rangle \rightarrow |1\rangle U|1\rangle$$

The resulting 4x4 matrix is

$$I \otimes U = \begin{bmatrix} u_{00} & u_{01} & 0 & 0 \\ u_{10} & u_{11} & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

Two-qubit gates



$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$

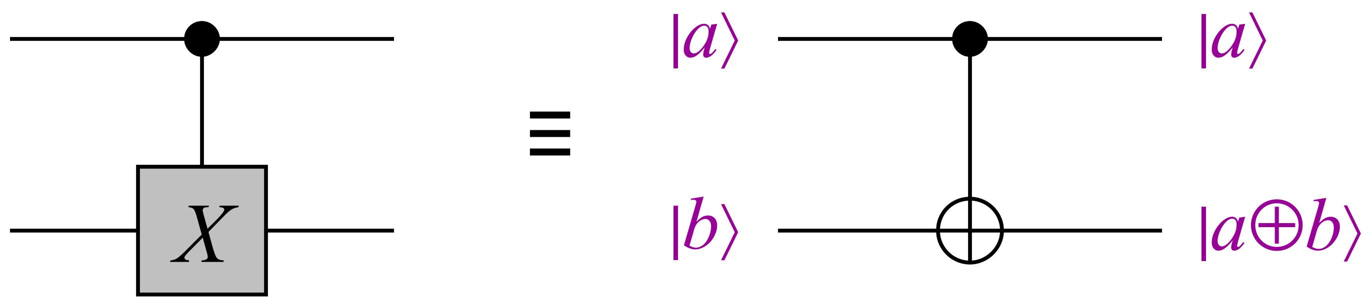
Resulting 4x4 matrix is controlled-U = C-U =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$$

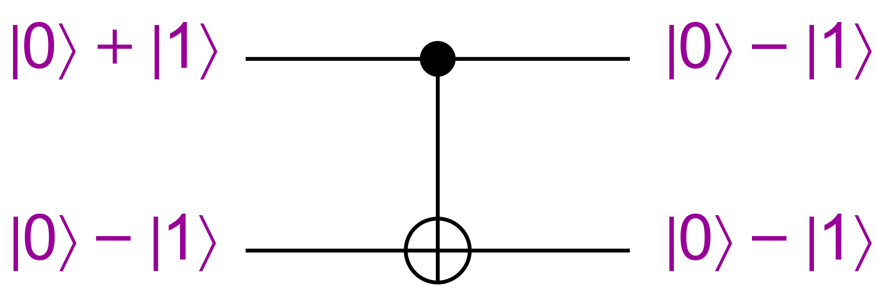
Maps basis states as:

$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$
$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle$
$|1\rangle|0\rangle \rightarrow |1\rangle U|0\rangle$
$|1\rangle|1\rangle \rightarrow |1\rangle U|1\rangle$

Controlled-NOT (CNOT)



Note: "control" qubit may change on some input states

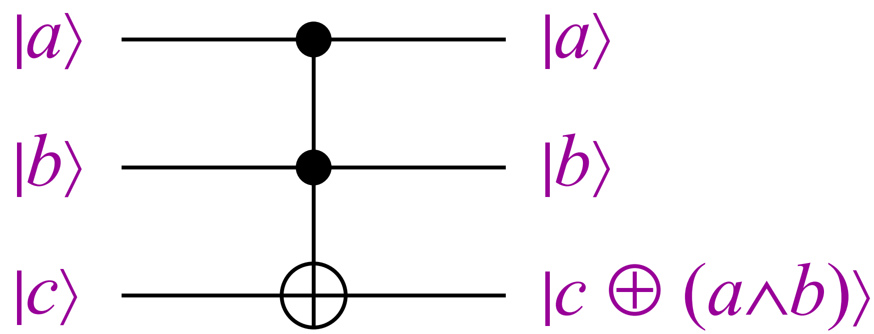


Universal sets of gates

Theorem: any unitary operation U acting on k qubits can be decomposed into $O(4^k)$ CNOT and one-qubit gates

Therefore, CNOT and all one-qubit gates are ***universal*** (classical analogue: AND and NOT gates)

Example: Toffoli gate
"controlled-controlled-NOT"



Can be simulated by CNOT, H, and $W = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

communication tasks

How much classical information in n qubits?

2^n-1 complex numbers are needed to describe an arbitrary n-qubit pure quantum state:

$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \ldots + \alpha_{111}|111\rangle$

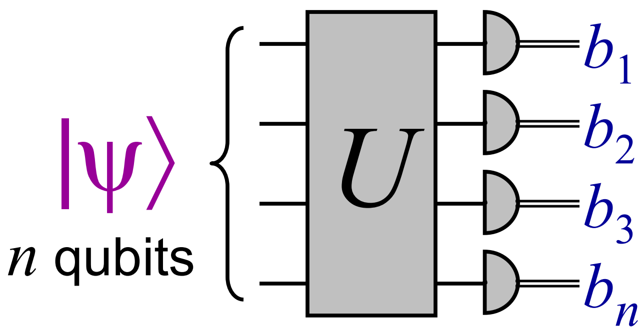
Does this mean that an exponential amount of classical information is stored in n qubits?

No! Holevo's Theorem [1973] implies: cannot convey more than n bits of information in n qubits

How much information does Nature have to store to maintain an n-qubit quantum state?

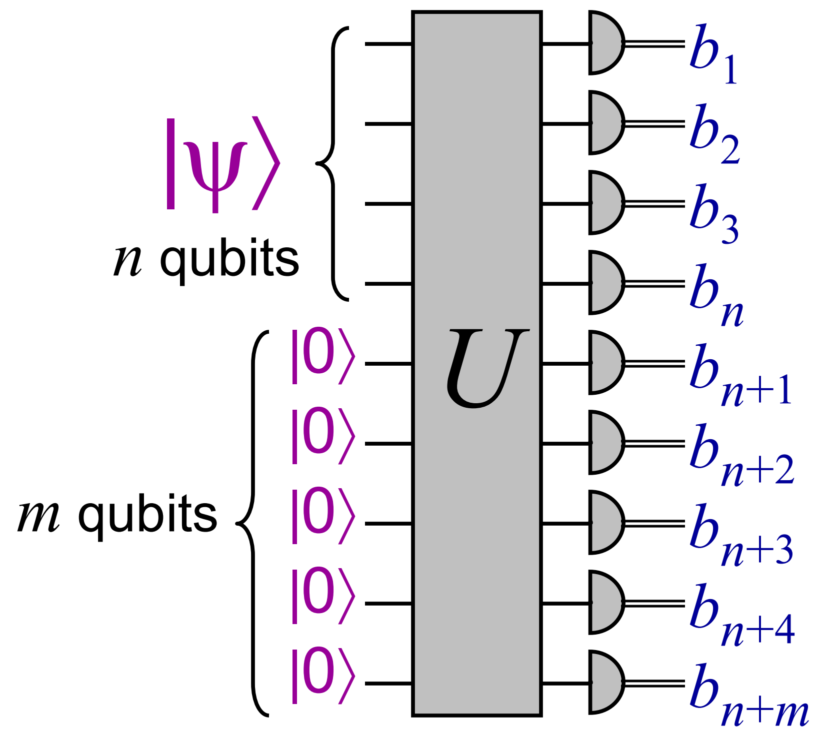
Holevo's Theorem

Easy case:



$b_1 b_2 \ldots b_n$ cannot convey more than n bits!

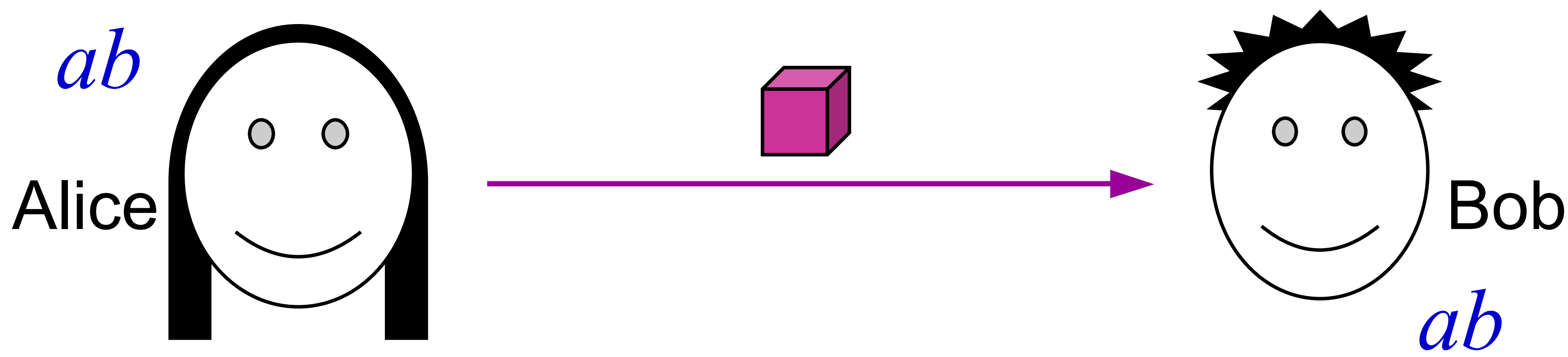
Hard case (the general case):



(proof is omitted here)

Superdense coding (prelude)

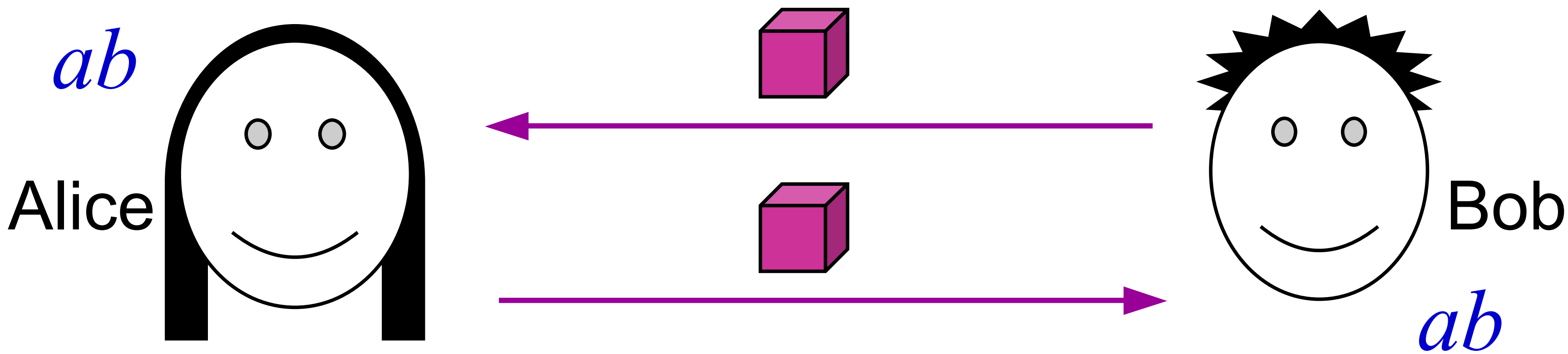
Suppose that Alice wants to convey two classical bits to Bob sending just one qubit



By Holevo's Theorem, this is ***impossible***

Superdense coding

In ***superdense coding***, Bob can send a qubit to Alice first



How can this help?

How superdense coding works

1. Bob creates the state $|00\rangle + |11\rangle$ and sends the ***first*** qubit to Alice

2. Alice: if $a = 1$ then apply X to qubit
   if $b = 1$ then apply Z to qubit
   send the qubit back to Bob

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

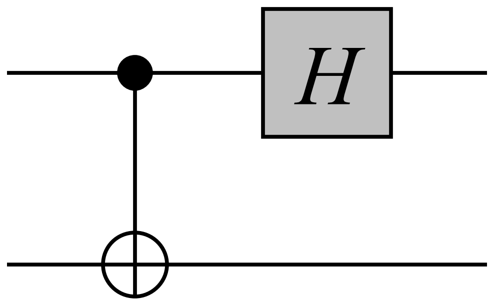
ab	state
00	$\lvert 00\rangle + \lvert 11\rangle$
01	$\lvert 00\rangle - \lvert 11\rangle$
10	$\lvert 01\rangle + \lvert 10\rangle$
11	$\lvert 01\rangle - \lvert 10\rangle$

(states: Bell basis)

3. Bob measures the two qubits in the ***Bell basis***

Measurement in the Bell basis

Specifically, Bob applies



input	output
$\lvert 00\rangle + \lvert 11\rangle$	$\lvert 00\rangle$
$\lvert 00\rangle - \lvert 11\rangle$	$\lvert 01\rangle$
$\lvert 01\rangle + \lvert 10\rangle$	$\lvert 10\rangle$
$\lvert 01\rangle - \lvert 10\rangle$	$\lvert 11\rangle$

to his two qubits ...

and then measures them, yielding ab

This concludes superdense coding

Review of partial measurements

Suppose one measures just the ***first*** qubit of the state

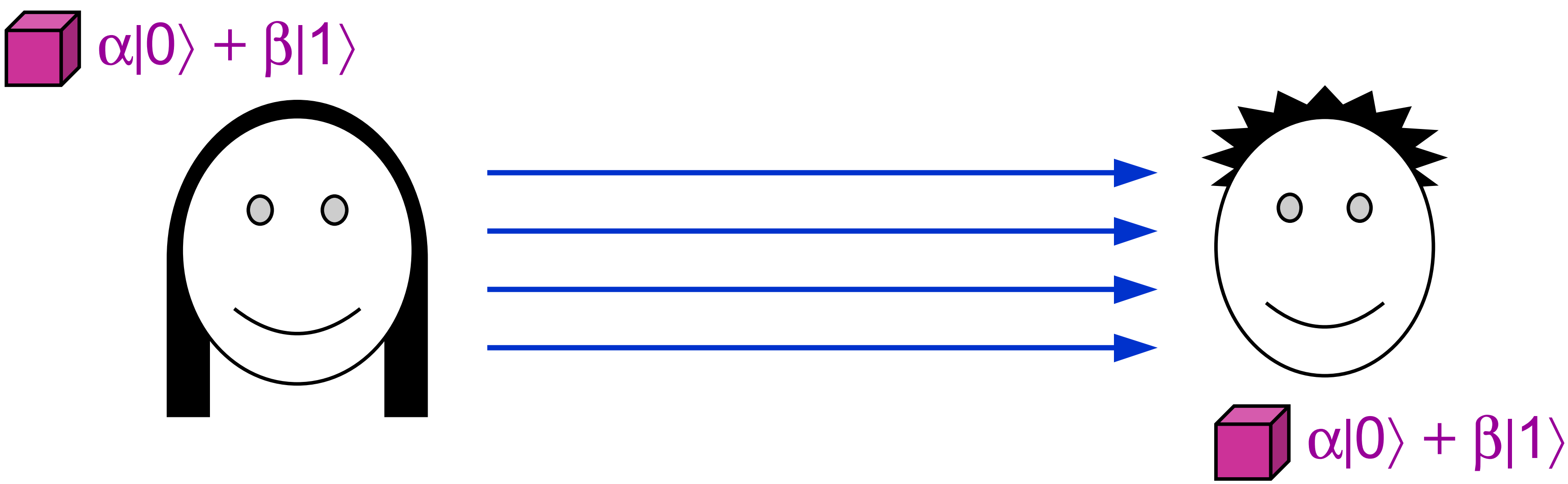
$$\frac{1}{2}|00\rangle + \frac{i}{\sqrt{3}}|01\rangle + \sqrt{\frac{5}{12}}|11\rangle = \sqrt{\frac{7}{12}}|0\rangle\left(\sqrt{\frac{3}{7}}|0\rangle + i\sqrt{\frac{4}{7}}|1\rangle\right) + \sqrt{\frac{5}{12}}|1\rangle|1\rangle$$

What is the result?

$$\begin{cases} 0, \quad \sqrt{\frac{3}{7}}|0\rangle + i\sqrt{\frac{4}{7}}|1\rangle & \text{with prob. } 7/12 \\ 1, \quad |1\rangle & \text{with prob. } 5/12 \end{cases}$$

Teleportation (prelude)

Suppose Alice wishes to convey a qubit to Bob by sending just classical bits

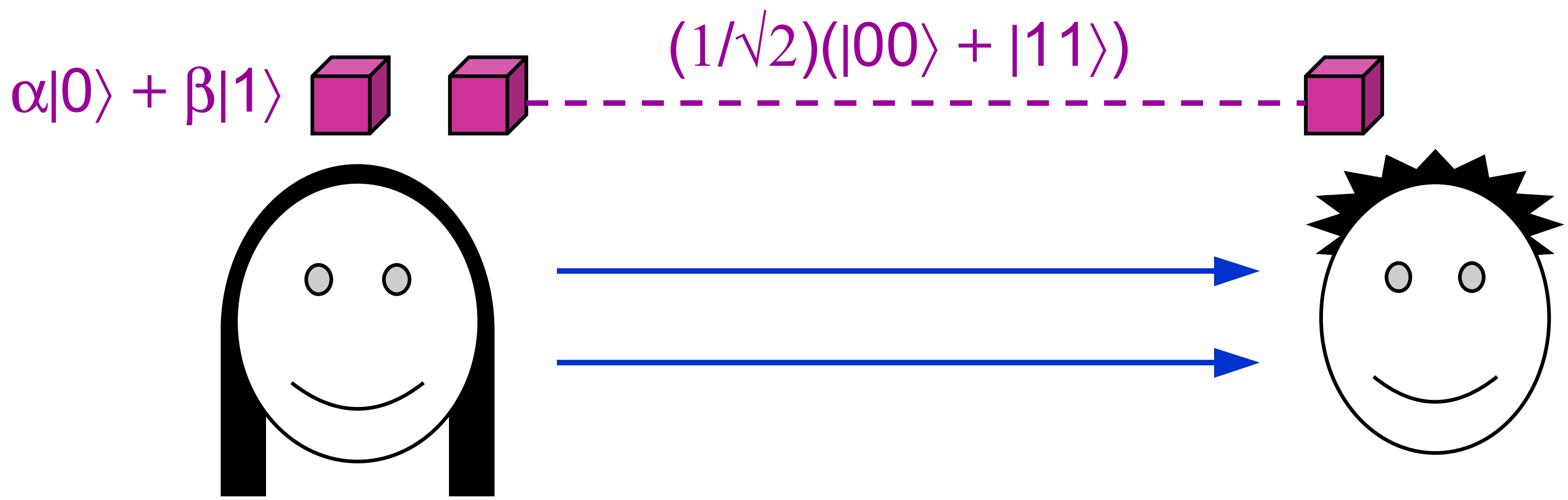


If Alice knows α and β, she can send approximations of them—but this requires infinitely many bits for perfect precision

Moreover, if Alice does ***not*** know α or β, she can at best acquire ***one bit*** about them by a measurement

Teleportation scenario

In teleportation, Alice and Bob also start with a Bell state



and Alice can send two classical bits to Bob

Note that the initial state of the three qubit system is:

$(1/\sqrt{2})(\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle)$

$= (1/\sqrt{2})(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

To be continued

Course announcements

- Send an email to qipcours@iqc.ca to be put on the course mailing list. We will normally only send information relevant to the course here (e.g. comments about the assignments).
- Office hours:
  - Carlos Perez (CS): Wednesdays 2:30-4:30pm
  - Donny Cheung (C&O): Thursdays 2:30-4:30pm
  - Marcus Silva (Physics): Fridays 10:00am-Noon ***or*** 1:00-3:00pm — ***is there a preference?***