Introduction to Quantum Information Processing

Lecture 7

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Overview of Lecture 7

- The one-out-of-four search problem
- The constant vs. balanced problem
- $H \otimes H \otimes \ldots \otimes H$
- Fourier sampling
- Preview of where black-box results are headed: period-finding
- Simulating black boxes

Query algorithms

Last time: quantum algorithm for computing $f(0) \oplus f(1)$ making just **1** query to f, whereas any classical algorithm requires **2** queries

This time: other, stronger quantum vs. classical separations, plus discussion about their relevance to algorithm design

one-out-of-four search

One-out-of-four search

Let $f: \{0,1\}^2 \rightarrow \{0,1\}$ have the property that there is exactly one $x \in \{0,1\}^2$ for which $f(x) = 1$

Four possibilities:

x	$f_{00}(x)$
00	1
01	0
10	0
11	0

x	$f_{01}(x)$
00	0
01	1
10	0
11	0

x	$f_{10}(x)$
00	0
01	0
10	1
11	0

x	$f_{11}(x)$
00	0
01	0
10	0
11	1

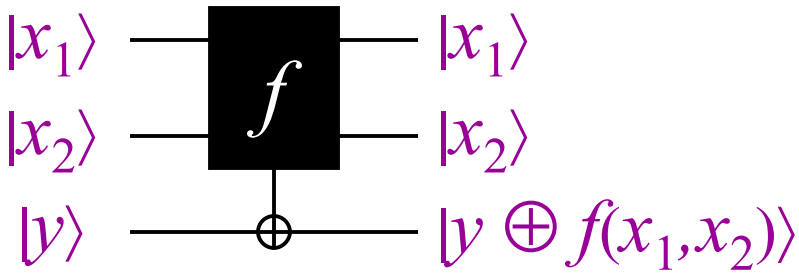
Goal: find $x \in \{0,1\}^2$ for which $f(x) = 1$

Classically: 3 queries are necessary and sufficient

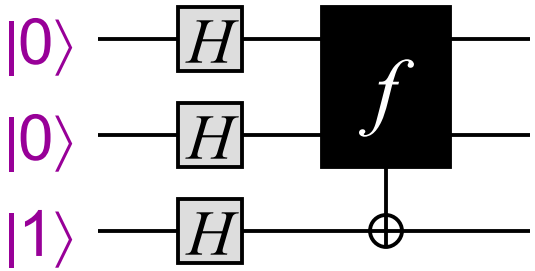
Quantumly: 1 query suffices!

Quantum algorithm

Black box for 1-4 search:



Start by creating phases in superposition of all inputs to f:



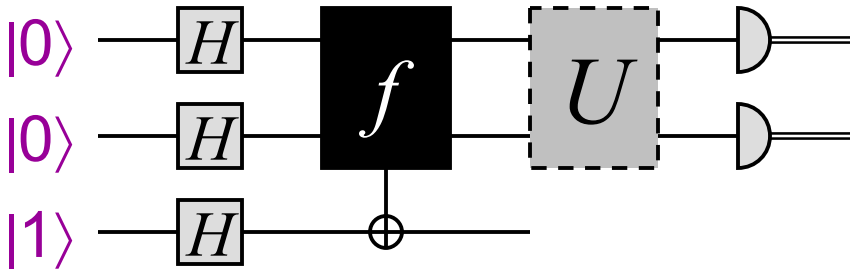
Input state to query:

$(|00\rangle + |01\rangle + |10\rangle + |11\rangle)(|0\rangle - |1\rangle)$

Output state:

$((-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle)(|0\rangle - |1\rangle)$

Quantum algorithm



where $U = \sum_{ab \in \{0,1\}^2} |ab\rangle\langle\psi_{ab}|$

Output state of the first two qubits in the four cases:

$|\psi_{00}\rangle = -|00\rangle + |01\rangle + |10\rangle + |11\rangle$

$|\psi_{01}\rangle = +|00\rangle - |01\rangle + |10\rangle + |11\rangle$

$|\psi_{10}\rangle = +|00\rangle + |01\rangle - |10\rangle + |11\rangle$

$|\psi_{11}\rangle = +|00\rangle + |01\rangle + |10\rangle - |11\rangle$

Note that these states are ***orthogonal!***

Challenge Exercise: simulate the above U in terms of H, Toffoli, and NOT gates

one-out-of-N search?

Natural question: what about search problems in spaces larger than ***four*** (and without uniqueness conditions)?

For spaces of size ***eight*** (say), the previous method breaks down—the state vectors will not be orthogonal

Later on, we'll see how to search a space of size N with $O(\sqrt{N})$ queries ...

constant vs. balanced

Constant vs. balanced

Let $f: \{0,1\}^n \rightarrow \{0,1\}$ be either constant or balanced, where

constant means $f(x) = 0$ for all x, or $f(x) = 1$ for all x

balanced means $\Sigma_x \ f(x) = \ 2^{n-1}$

Goal: determine whether f is constant or balanced

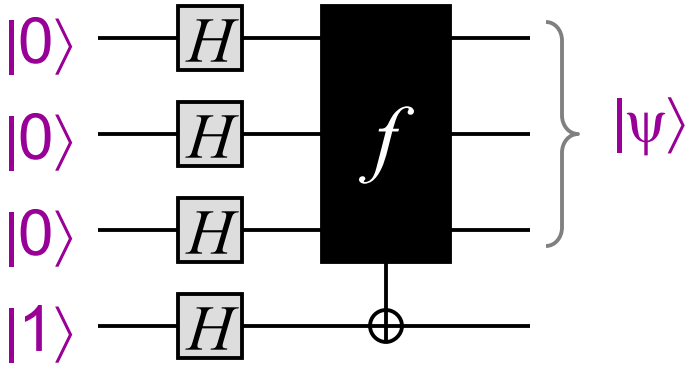
Classically: $\frac{1}{2}\ 2^n + 1$ queries are needed

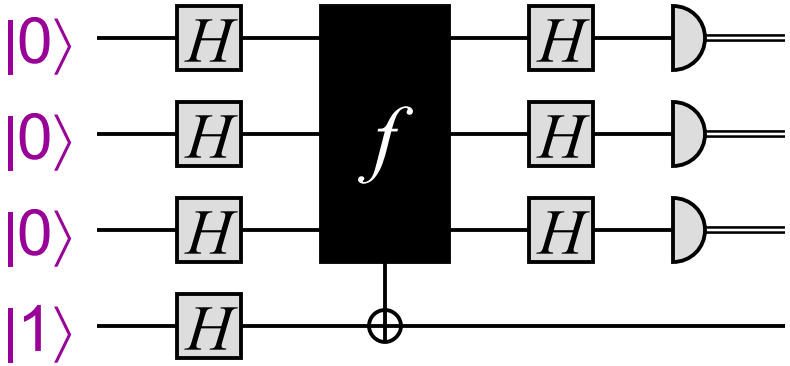
Example: if $f(0000) = f(0001) = f(0010) = \ldots = f(0111) = 0$ then could be either

Quantumly: just **1** query suffices!

[Deutsch & Jozsa, 1992]

Quantum algorithm





Constant case: $|\psi\rangle = \pm \sum_x |x\rangle$ (why?)

Balanced case: $|\psi\rangle$ is ***orthogonal*** to $\pm \sum_x |x\rangle$ (why?)

How to distinguish between the cases? What is $H^{\otimes n}|\psi\rangle$?

Constant case: $H^{\otimes n}|\psi\rangle = \pm |00...0\rangle$

Balanced case: $H^{\otimes n}|\psi\rangle$ is orthogonal to $|0...00\rangle$

Last step of the algorithm: if the measured result is 000 then output “constant”, otherwise output “balanced”

Probabilistic *classical* algorithm solving constant vs balanced

But here's a classical procedure that makes only **2** queries and has ***one-sided*** error probability ½ :

1. pick $x_1, x_2 \in \{0,1\}^n$ randomly
2. **if** $f(x_1) \neq f(x_2)$ **then** output balanced **else** output constant

If f is constant the algorithm always succeeds

If f is balanced the algorithm succeeds with probability ½

By repeating the above procedure k times:
$2k$ queries and one-sided error probability $(½)^k$

Therefore, for large n, $<< 2^n$ queries are likely sufficient

$$H \otimes H \otimes \ldots \otimes H$$

About $H \otimes H \otimes \ldots \otimes H = H^{\otimes n}$

Theorem: for $x \in \{0,1\}^n$, $H^{\otimes n}|x\rangle = \frac{1}{2^{n/2}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$

where $x \cdot y = x_1 y_1 \oplus \ldots \oplus x_n y_n$

Example: $H \otimes H = \frac{1}{2} \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}$

Pf: For all $x \in \{0,1\}^n$, $H|x\rangle = |0\rangle + (-1)^x |1\rangle = \sum_y (-1)^{xy} |y\rangle$

Thus, $H^{\otimes n}|x_1 \ldots x_n\rangle = \left(\sum_{y_1} (-1)^{x_1 y_1} |y_1\rangle\right) \ldots \left(\sum_{y_n} (-1)^{x_n y_n} |y_n\rangle\right)$

$= \sum_y (-1)^{x_1 y_1 \oplus \ldots \oplus x_n y_n} |y_1 \ldots y_n\rangle$ ∎