18.419 Random Walks and Polynomial-Time Algorithms	April, 25

Lecture 18. Hit-and-run

Lecturer: Santosh Vempala *Scribe: Mikhail Alekhnovitch*

Assume that a convex body K is given and the goal is to sample a random point in K. This lecture is dedicated to the following random walk called "Hit-n-run":

- While at a point $u \in K$, choose a random line L through u
- Go from u to y picked uniformly from $L \cap K$
- Repeat

Analytical density.

Our first goal is to calculate the probability density $\rho(u, x)$ of making a step from u to x. To warm up consider the case when K is the unit ball with u in the center. Then the transition density is propotional to the $(n-1)$-dimensional volume of the sphere with radius $|x - u|$:

$$\rho(u, x) \propto \frac{1}{\text{Vol}_{n-1}(S_n(|x-u|))} \propto \frac{1}{|x-u|^{n-1}}$$

Denote by $\pi_n = \text{Vol}(B_n)$ and $n\pi_n = \text{Vol}_{n-1}(S_n)$ the volume of n-dimensional unit ball and n-dimensional sphere resp. Then $\rho(u, x)$ is given by the formula

$$\rho(u, x) = \frac{1}{n\pi_n} \frac{1}{|x-u|^{n-1}} \tag{1}$$

One way to see that the constant equals $1/(n\pi_n)$ is to check that the integral of $\rho(u, x)$ over B_n equals to 1:

$$\int_{B_n} \frac{1}{n\pi_n} \frac{dx}{|x-u|^{n-1}} = \begin{pmatrix} |x-u| = r \\ dx = r^{n-1} dr \cdot dw \end{pmatrix} = \int_{S_n} \frac{dw}{n\pi_n} \int_0^1 \frac{r^{n-1} dr}{r^{n-1}} = 1.$$

In the general case, the transition density equals to the probability density of choosing the line L_{ux} going through x multiplied by the probability of choosing x on this line. Denote by $\ell(u, x)$ the length of the intersection $L_{ux} \cap K$. In case of the unit ball $\ell = 2$, thus in general the density is given by

$$\rho(u, x) = \frac{2}{n\pi_n} \frac{1}{\ell(u, x)|x-u|^{n-1}}$$

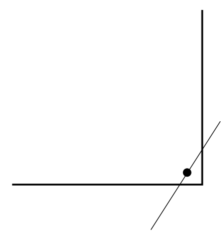

Figure 1:

Obviously, $\rho(u, x) = \rho(x, u)$ and the uniform distribution π is stationary.

Step size.

Our next consideration is the average step size. In the bad case, when u is close to a "corner" with high probability the chosen line L will intersect the "sides" of the corner and the expected step size is exponentially small (see fig. 1). However, if K contains a ball of sufficiently large radius r centered in u then the step size is big. It is convenient to define the following technical notion: let $F(u)$ be the step size given by

$$\Pr\Big(|u - x| \le F(u)\Big) = \frac{1}{8}.$$

If K contains a ball of raduius r then $F(u) \ge r/8$. Note (from the previous lecture) that on average $F(u) \ge \frac{1}{\sqrt{n}}$.

Conductance.

As usual, we define the conductance as

$$\Phi = \inf_{0<Q(S)<\frac{1}{2}} \frac{\Phi(S)}{Q(S)} = \inf \frac{\int P_u(\bar{S})du}{Q(S)}.$$

As in the ball random walk, we need the following two components:

1. isoperimetry
$$\mathrm{Vol}(K \setminus S_1 \setminus S_2) \ge \frac{d(S_1, S_2)}{D} \cdot \frac{\mathrm{Vol}(S_1)\mathrm{Vol}(S_2)}{\mathrm{Vol}(K)}.$$
2. if $|u - v|$ is small then the variation distance $|P_u - P_v| < 1 - c$.

As we show in the next example, the second property doesn't hold under the Euclidean distance. The points u, v and x on figure 2 are very close to each other, however the probability $\rho(u, x)$ differs drastically from $\rho(v, x)$. Thus, it is not usefull to bound the

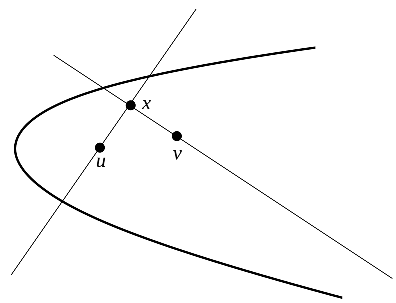

Figure 2:

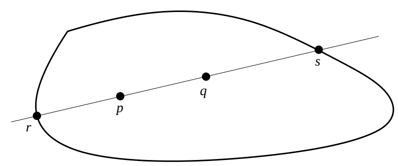

Figure 3:

Euclidean distance, we shoud use some more adequate metric, that takes into account the geometry of K. One example of such metric, is the following cross-ratio distance.

For two points $p, q \in K$ let r and s be the points of the intersection of line L_{pq} with K so that r is closer to p and s closer to q resp. (see figure 3). Let

$$d_K(p,q) = \frac{|p-q||r-s|}{|r-p||q-s|}$$

be the new distance, which will lie in the heart of our analysis. Clearly

$$d_K(p,q) \geq \frac{|p-q|}{\min(|r-p|,|s-q|)} \geq 4\frac{|p-q|}{D}.$$

Does $d_K(p,q)$ have an isoperimetric inequality? As we will show in the next lecture

$$\mathrm{Vol}(K \setminus S_1 \setminus S_2) \geq d_K(S_1,S_2)\frac{\mathrm{Vol}(S_1)\mathrm{Vol}(S_2)}{\mathrm{Vol}(K)}.$$

Why does this distance correlate with probabilistic distance? The answer is given by the following

Lemma 1. *For any $u, v \in K$ s.t. $d_K(u,v) < \frac{1}{8}$ and $|u-v| \leq \frac{2}{\sqrt{n}}F(u)$ holds*

$$|P_u - P_v| \leq 1 - \frac{1}{500}.$$

Proof. Recall, that $|P_u - P_v| = \sup_A P_v(A) - P_u(A)$. We want to identify "good" points x for which $P_v(x) \geq \Omega(P_u(x))$. Let

$$A_1 = \{x \in A|\ |x - u| \leq F(u)\}$$

be the first set of exceptional points. By the definition of $F(u)$, $P_u(A_1) \leq \frac{1}{8}$. Let

$$A_2 = \{x \in A|\ |(x-u)^T(x-v)| \geq \frac{1}{\sqrt{n}}|x-u||u-v|\}.$$

These are points x for which the vector $x - u$ is "far" from orthogonal to $u - v$ (see figure 4). A straightforward integration of the spherical surface shows that $P_u(A_2) \leq \frac{1}{6}$.

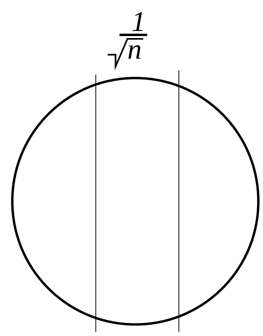

Figure 4:

Let us denote by $a(u,x)$ and $a(x,u)$ the points of the intersection of L_{xu} that are closer to u and x resp. (see figure 5). Define the third exceptional set as

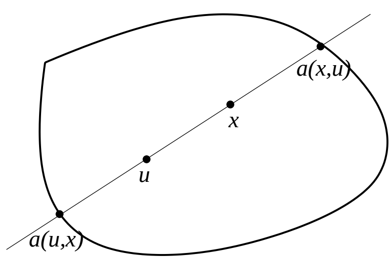

Figure 5:

$$A_3 = \{x \in A|\ |x-u| \leq \frac{1}{3}|u - a(u,x)|\}.$$

Clearly $P_u(A_3) \leq \frac{1}{3}$. Thus the overall probability of the exceptional steps is bounded by $\frac{1}{8} + \frac{1}{6} + \frac{1}{3} = \frac{3}{4}$ and it is left to consider the case when none of them occurs. Otherwords we have to show that for all points $x \in K \setminus A_1 \setminus A_2 \setminus A_3$ holds

$$|x-u| \approx |a-v|$$
$$\ell(x,u) \approx \ell(x,v).$$

The first is easy:

$$\begin{aligned}
|x-v|^2 &= |x-u|^2 + |u-v|^2 + 2(x-u)(u-v) \\
&\overset{(x\notin A_2)}{\leq} |x-u|^2 + |u-v|^2 + \frac{2}{\sqrt{n}}|x-u|\cdot|u-v| \\
&\overset{(x\notin A_1, |u-v|\leq F(u)/\sqrt{n})}{\leq} |x-u|^2 + \frac{4}{n}|x-u|^2 + \frac{4}{n}|x-u|^2 \\
&= (1+\frac{8}{n})|x-u|^2.
\end{aligned}$$

Let's do the second. For that we will use a geometric fact from the high school, called Menelaus Theorem. We need the following geometric set-up, depicted on figure 6. Let $y = a(u,v)$, $z = a(v,u)$, $p = a(u,x)$, $q = a(x,u)$, $r = a(v,x)$ and $s = a(x,v)$. Let p' be the intersection of the segments $[u,p]$ and $[y,r]$.

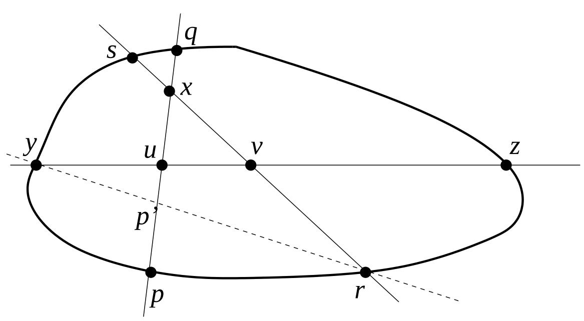

Figure 6:

By Menelaus Theorem,

$$\frac{|x-r|}{|v-r|} = \frac{|u-y|}{|v-y|} \cdot \frac{|x-p'|}{|u-p'|}.$$

We have

$$\frac{|u-y|}{|v-y|} = 1 - \frac{|v-u|}{|v-y|} > 1 - d_K(u,v),$$

which implies

$$\begin{aligned}
\frac{|x-v|}{|v-r|} &= \frac{|x-r|}{|v-r|} - 1 \geq (1-d_K(u,v))\frac{|x-p'|}{|u-p'|} - 1 \\
&= \frac{|x-u|}{|u-p'|}\left(1 - d_K(u,v)\frac{|x-p'|}{|x-u|}\right) > \frac{1}{2}\frac{|x-u|}{|u-p'|}.
\end{aligned}$$

(Here we used that $|x-p'| < 4|x-u|$ since $x \notin A_3$). Hence

$$|v-r| < 2\frac{|x-v|}{|x-u|}|u-p'| \leq 2\frac{|x-v|}{|x-u|}|u-p|.$$

Similarly,

$$|v-s| < 2\frac{|x-v|}{|x-u|}|u-q|$$

and

$$\ell(x,v) < 2\frac{|x-v|}{|x-u|}\ell(x,u) \leq (1+\frac{4}{n})\ell(x,u).$$

□