Crypto-complexity Based Models of Efficiency in Capital Markets*

(Extended Abstract)

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Abstract

Investment decisions pose interesting combinatorial optimization problems for computer scientists, and are vital to the operation of investment markets. We study capital markets in the context of computer science theory. We apply crypto-complexity techniques to model and furnish proofs for fundamental properties in capital markets. This marks the first known application of crypto-complexity theory outside the realm of pure computer science problems.

We are interested in understanding the correlation between true value and price of securities in the investment markets. An *efficient capital market* is one in which the prices of securities accurately reflect their objective value, and any new information that effects the value of a security is instantly incorporated into its price. Determining efficiency, or lack thereof, in capital markets is a key issue in economics and finance. The concept of efficiency is crucial to investors who compare dynamic strategies to "buy-and-hold" strategies, and off-set their risks by diversifying their investments. The direct implication of efficiency in a capital market is that no investor can consistently beat that market. Previously, the central bottleneck for rigorous proofs in economics and finance literature on efficient capital markets was the lack of a proficient model for investment markets, and in particular one that provides a sufficiently robust notion of "information" to fully describe limitations of investor strategies. The following are our main contributions:

- *We devise a family of very robust commodity models.* These models emulate investment markets by time-dependent flow of streams in a network. We allow for a wide variety of blended rules to incorporate diverse investment options (like selling short, buying futures, deferring investment decisions), and various features (like dynamic adjustment and price retention).
- *Our new models provide simple and precise paradigms which bear essential characteristics of a wide range of markets, yet lend themselves to rigorous analysis.* Our models are new and interesting in their own right because they pose challenging combinatorial optimization problems.
- *The flow in the network forms pseudo-random time sequence with respect to various cryptographic complexities.* Investors emulate investment decisions (buy, sell, buying futures, short selling, etc.) by apportioning their withdrawals from the flow, at certain edges in the network model, with the objective of maximizing their returns using available information.
- *Our major contributions, besides proposing feasible models, are rigorous proofs for relating efficiency to predictability in our models of investment markets.*
- *We show that investors can gain leverage from short-selling, and they would be able to extract all the predictability from our market model in a single swoop*, and not leave anything for the subsequent investors to exploit.
- *Our analysis can be applied to analyze other markets.* For example, we show that our barter computer resources model is analogous to multiuser and distributed systems. This promises useful applications of our work in studying distributed systems.

For the convenience of the referees, we have included a set of appendices with detailed proofs of all the theorems.

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1 Introduction

1.1 Previous Work

We describe new models for capital markets that are based on combinatorial optimization of investment portfolios for pseudo-random price sequences. Our work will interest computer scientists because we introduce crypto-complexity techniques to a new area, and provide fresh perspectives on combinatorial optimization issues.

We are interested in understanding how investors can optimize their investments, and how their investment decisions effect the relation between true value and price of securities in the investment market. We analyze the combinatorial optimization of investment portfolios in relation with determining *efficiency* in capital markets with pseudo-random variations in price: A capital market is classified as *efficient* if the price of securities accurately reflects their true value. The combinatorial optimization of investment portfolios and the concept of efficiency are crucial to investors who compare dynamic strategies to "buy-and-hold", and off-set their risks by diversifying their investments. For examples of capital markets, see § 2.2 and § 2.3.

Our two-parameter stream model is an enhancement and extension of the classical one-period, two-parameter portfolio model due to Markowitz [12, 13]. The central issues in classical two-parameter model are expected return, risk potential, market equilibrium, and correlation between $\tilde{R}_q$ and $\tilde{R}_s$. We refer the reader to Chapters 7 through 9 of Fama's book on security markets [8] for a concise exposition of issues related to the two-parameter model. For more detailed discussions, the reader is referred to the works of Markowitz [12, 13], Merton [14], Sharpe [18], Lintner [11], Brennan [6], and Black [2]. Empirical test of the two-parameter model have been performed by Fama and Macbeth [9], Miller and Scholes [15], and Black et al. [3]. The Markowitz model is limited to two parameters and one-period, and it lacks sufficiently robust notion of "information" to fully describe limitations of investor strategies, which is a central bottleneck for rigorous proofs in economics and finance literature on efficient capital markets.

We devise several robust commodity network models for investment markets which afford very powerful and useful paradigms of information for rigorous proofs. We exploit crypto-complexity tools, introduced by M. Blum and Micali [4, 5], and Shamir [16, 17], to model relative information, and relate efficiency to combinatorial optimization of investment decisions.

The primary challenges particularly relevant to computer scientists are:

- Develop models of efficient capital markets that are realistic with respect to the actual security markets, yet lend themselves to rigorous analysis. Subsequently, the key problem is to prove efficiency for these models. We also need to explore the salient properties which induce efficiency in these models, and then identify these properties in actual capital markets.
- Analyze the predictability of a capital market with respect to computational devices of varying computational power. For example, there maybe certain trends which cannot be predicted at all, other trends that lend themselves to timely and accurate prediction on a supercomputer (like CRAY-YMP), and yet other trends which require much less computational power (like a lap-top) to forecast.

1.2 Viewing Price Sequences as Pseudo-random Sequences

We analyze the time series data of capital market prices as pseudo-random sequence with respect to the cryptographic complexity required to predict them. This original and novel approach of ours enables us to apply crypto-complexity tools from theoretical computer science to study capital markets, and introduce several fresh perspectives in analyzing combinatorial optimization of investment decisions.

The prices in capital markets can be viewed as a time series data: The price of a particular security (or some market index or indicator like interest rates) can be analyzed as a time sequence of random variables $\{\tilde{f}_t\}$.[1] If one were to study trends in instantaneous prices then it would be continuous time series. However, in practice, we can break capital market data into a discrete time series, recorded after

[1] Following the standard notation of finance, a ˜ notation above a variable indicates that it is a random variable.

a fixed amount of transactions, or based on daily closing prices (possibly using Friday's closing prices for Saturday and Sunday as well) without any substantial hindrance to the significance to our analysis.

Studying trends is related to the problem of forecasting $\tilde{f}_{k+1}$ based on the knowledge of $\{\tilde{f}_1, \cdots, \tilde{f}_k\}$. Let each $\tilde{f}_t$ have some apriori probabilistic distribution Δ. Let $\mathcal{C}(k)$ be a complexity measure. We define $\mathcal{C}(k)$-predictability as follows:

Definition 1.2.1 ($\mathcal{C}(k)$-predictable) *A cryptographic sequence $\{\tilde{f}_t\}$ is $\mathcal{C}(k)$-predictable, if one can always predict $\tilde{f}_{k+1}$ exactly, based on the knowledge of $\{\tilde{f}_1, ..., \tilde{f}_k\}$ using a computational device with computational power at most $\mathcal{C}(k)$, for some complexity class $\mathcal{C}(k)$.*

We observe that this definition of cryptographic sequences is similar to the classical definitions of Manuel Blum and Silvio Micali [4, 5], and Adi Shamir [16, 17]. The fundamental classical research on pseudo-random sequences is primarily concerned with predicting the next bit in a sequence of bits. The major relevance to our work is the time series in which each element is represented by a finite number of bits. We refer the reader to the above papers [4, 5, 16, 17] for a review of classic cryptographic complexity.

The sequence of prices, $\{\tilde{f}_t\}$, consists of a scalar $\tilde{f}_t$ at each instant of a time base. Each $\tilde{f}_t$ maybe partionable into multiple quantities. A multiple time series usually includes a function which relates the various elements, and may be as simple as an element-wise sum of several sequences, or some complicated formula involving more intricate mathematical operations. For example, a sequence $\{\tilde{f}_t\}$ can determined by a element-wise sum of three sequences, $\{\tilde{a}_t\}, \{\tilde{b}_t\}, \{\tilde{c}_t\}$, i.e. $\tilde{f}_t = \tilde{a}_t + \tilde{b}_t + \tilde{c}_t$. Predicting one component might not as easy/difficult as another component. For instance, $\tilde{a}_t$ can be truly random (i.e. completely unpredictable), $\tilde{b}_t$ can be predicted using a computational device with a at least $\mathcal{C}(k)$ power, and $\tilde{c}_t$ is (pseudo-)random with respect to $\mathcal{C}(k)$ power.

As a more realistic example, consider the retail price of gas at time t (denoted by $\tilde{f}_t$). Suppose $\tilde{f}_t$ is determined by three factors: (1) Wholesale price of oil ($\tilde{w}_t$). (2) Quantified effect of brand-name of the gasoline ($\tilde{b}_t$). (3) Quantified effect of location of the gasoline station ($\tilde{l}_t$).

In other words, the sequence $\{\tilde{f}_t\}$ is determined by a three sequences, $\{\tilde{w}_t\}, \{\tilde{b}_t\}, \{\tilde{l}_t\}$, and by some relationship like $\tilde{f}_t = \tilde{w}_t + \tilde{b}_t + \tilde{l}_t$. It should be noted that, in general, the complexity of predicting different components will be different. For example, the effect of the wholesale price is easiest to assess and quantify, the effect of the brand-name is slightly harder to assess and quantify, and the the effect of the location is hardest to forecast and quantify. The diversity in the computational complexity of predictability for different components are neatly incorporated into our model.

Our model does not restrict all investors to equivalent power when it comes to information and prediction power. In fact, every investor can have its own information base, and use it to predict future prices with respect to its computational constraints. This implies that some components that are predictable to one investor, will not appear predictable to other investors. This feature models a vital property of the investment markets: Forecasts of various investment companies and different investors do not necessarily have to be the identical or equally reliable.

We use sequences of random variables, $\{\tilde{f}_t\}_t$ to record the values in a sequence (at time indexed by t). Elements in the sequences may be sums of random variables. For example, if the total flow at time t ($\tilde{f}_t$) consists of m components ($\tilde{g}_{1,t}, ..., \tilde{g}_{m,t}$), then: $\tilde{f}_t = \sum_{k=1}^{m} \tilde{g}_{k,t}$, where $\{\tilde{g}_{k,t}\}_t$ are cryptographic sequences in their own right for all values of k between 1 and m inclusive. If a particular player knows the exact value of one of the components, it may substitute that precise value for the corresponding random variable.

1.3 Essential Ideas of Our Family of Models

We present a family of new models. Our models emulate investment markets by flow of time-dependent streams in a network, which forms pseudo-random time sequence with respect to cryptographic complexities. These models easily simulate various characteristics from a diverse range of markets, yet they prove conducive to rigorous analysis and detailed explanation. The specialized blend of rules governing each model allows the investors to emulate a wide variety of investment options (like selling short, buying futures, deferring investment decisions) by apportioning their rights to withdraw certain fraction of flow at designated edges in the network model (with the objective of maximizing their returns using available information). More sophisticated options, like dynamic adjustment and price retention, are also built into our models. See § 3.1 and § 3.2. (For details, see Appendices B.2 and C.2).

Our most sophisticated model is a network of time-dependent streams of commodity prices in a network with three different kinds of nodes, each node serves one of the functions from merger, split, and return. See § 5 for description of the network. (Technical details are can be found in Appendices B.4 and F). Figure 1 illustrates a network with all these nodes in action. The investors apportion their withdrawal rights (5% in the figure) so as to maximize its profit from the return nodes by extracting all its rights from the stream with greater flow. The figure also exhibits a merger in which two flows (of 95 each) are channeled together into one combined flow (of 190), and a split which partitions a flow 360 into two streams with flows of 200 and 160 (respectively, according to split parameter 0.625). Our model exhibits a time-space duality, in the sense that the progress along the horizontal axis represents time. Also observe that an investor may move twice, as is done by Investor 1 in Figure 1.

We model several different circumstances in which the prices of securities efficiently reflect their true value. Our analysis can be readily applied to analyze a wide variety of capital market models. We also successfully eliminate the necessity for intermediums like money since our commodity models can simulate investors trading rights to invest in a security for either (i) rights to invest in the same security at a different instant, or (ii) rights to invest in another security at the same instant, or (iii) rights to invest in another security at a different instant.

We use the analogy of water streams to explain our model. Farmers, living in a valley surrounded by several streams, face the choice of scheduling their only water pump between accessible streams, so as to maximize their water acquisition with respect to the information available to them. These farmers may also have the choice of deferring their water access rights over time, or the opportunity to tap their rights at an earlier date. (See § 3). Of course, in general, water can be substituted by any commodity. See § 2 and Figure 1 for several other examples (like wheat, oil, multiuser computer, systems, etc.).

We first model the investment market by two commodities with prices represented by two time-dependent streams of values. The basic two-parameter stream model with 11 farmers is illustrated in Figure 2. The two-parameter stream model consists of a row of several farmers stationed in a valley banking on two streams. Each farmer has rights to withdraw a fraction α of water from the stream it chooses to tap, where α corresponds to an investor's investment resources in capital markets. A farmer has to choose an apportioning strategy by selecting an x such that αx fraction will be withdrawn from Stream A, and $\alpha(1-x)$ fraction will be withdrawn from Stream B. Optimal strategy selects x to maximize the amount of water acquired. Our models are innovative, yet based on classical two-parameter models of Markowitz [12, 13]. They afford a sophisticated modeling of information, introduce crypto-complexity techniques, and combine computer science theory techniques with economically realistic ones.

Suppose, Stream A's level is 100 units, Stream B's level is 97 units, and α (which corresponds to a farmer's resources) is 3%. The farmer should optimally dedicate his water pump to Stream A by choosing $x = 1$. Consequently, in the process of extracting 3 units, the farmer would have drained Stream A's level to 97 units, and left Stream B's level unchanged at 97 units. In this situation, it takes one farmer to equalize the level of the two streams. We show that sufficient number of farmers can equalize any disparity in the $\mathcal{C}(k)$-predictable components of the streams, and consequently we prove market efficiency. Furthermore, if short-selling is allowed, the disparity can be removed by one farmer alone.

1.4 Overview

§ 2 highlights the importance of efficiency in capital systems using wheat commodity market as an example. It also presents a novel perspective by viewing computer systems as investment market, and suggests direct applications of our work in computer science. § 3 thoroughly explains the two-parameter multiplayer model. § 4 analyzes the RETURN node which is most crucial to the extension of this model to the network model. We can describe the two-parameter stream model as the network shown in Figure 3. § 5 briefly describes the network model and summarizes the main results. (We leave formal proofs to Appendix F due to lack of space.) (Formal proofs are furnished in § F.)

The two-parameter model is extended by allowing the parameters to be substituted by portfolios, and thereby allowing multiple parameter portfolios. Similarly, the two-parameter stream model can be extended by allowing the streams to be substituted by sets of streams to give us multiple parameter stream model (by recursion), as indicated in § 4.1 (see Appendix E for further details). We also introduce and exploit some crypto-complexity tools because, in general, an investor is unable to gauge the exact value of a security (using its analytical capabilities), and analogously, a farmer in our model is unable to gauge the exact level of the stream. The investors (and farmers) use their estimates for the values of securities (and levels of streams) to formulate their strategy. (See § 4.2.)

2 Importance of Efficiency: Examples of Investment Markets

2.1 Role of Efficiency in Investment Markets

The following two questions about the investment markets are important to investors, brokers, and academicians:
(1) Does the investment market efficiently reflect the true value of the securities, or can an individual "beat the market" by consistently identifying under-priced stocks to buy, and identifying over-priced stocks to sell?
(2) Is it possible to forecast (with any appreciable confidence) the future stock prices by observing past trends, or do the prices follow a random variation, and are virtually unpredictable?

We are primarily interested in understanding the correlation between true value and price of securities in the investment market at a given time, and the role that this correlation plays in the combinatorial optimization of the investment portfolio. We also want to gauge the validity of forecasting future price of securities based on the past trends in its price. In this pursuit, we study the behavior of predictable elements, and analyze their role in determining prices.

Efficient capital market was a term coined by Fama [7] to describe an investment market which possesses the characteristic that security prices completely reflect all available information. A key issue concerning economics and finance is: "Given a capital market, can we determine if it is *efficient*?" The direct implication of efficiency in capital markets is that no investor can consistently beat the market.

The *efficient capital market* theory has had a profound impact on the investment industry. It may be that certain capital markets are essentially efficient, but some knowledgeable investors attempt to use their expertise to exploit any holes in the efficiency.This motivates us to introduce the notion of relative pseudorandomness: What appears random to casual investors may not appear random to investment corporations (and some professional investors) with certain knowledge, experience, and tools (like computer resources) for market analysis.[2] In other words, a limited amount of "inefficiency" exists in certain capital markets, but only the sagacious investors can capitalize on this "inefficiency".This would practically explain the ability of investors with close access to the market (and, in certain cases, analytical tools and substantial computing power, like CRAY-YMP or Connection Machines) to consistently gain profit from the market, by various maneuvers (like arbitrage). Yet the market appears efficient and unpredictable to ordinary investors. The reader will observe this virtual difference of market perception between different classes of investors is incorporated as a key aspect in our models. One would conclude

[2] In fact, forecasting the future security prices is a separate problem, and was first thoroughly examined in 1900 CE by Louis Bachelier in his dissertation [1]. Bachelier's failure to find any appreciable correlation between various movements of the stock market coupled with his inability to explain erratic variations in stock prices, sparked the *Random Walk* theory of stock market, which explains price fluctuations as random variations. This theory dismisses the possibility of forecasting future security prices with any appreciable accuracy and confidence. The *Random Walk* theory was given more credulity fifty-three years later when Maurice Kendall [10] was unable to find any pattern in his analysis of the weekly changes in nineteen stock prices on the London stock exchange. A standard informal explanation (but not a proof) for the *Random Walk* theory is that stock prices adjust very quickly to new data and available information. The sporadic occurrence of major events coupled with rapid reaction in adjusting to these events make it impossible for most players to take advantage of any major events. The main contention of the *random walk* theory is that: No player can beat the market consistently, because there is no appreciable trend in the time sequence of stock prices.

that professional investment strategists are as unlikely to appreciably outperform the market as are ordinary investors if the predictable component is relatively inconsequential.[3]

2.2 Example of Wheat Commodity Market and Importance of Efficiency

In this subsection, we will further motivate the importance of efficiency in capital markets by using the example of the wheat market. In general, producers, consumers, and investors protect themselves by diversifying their investments in the market in order to balance profits with risks. Efficiency is essential to all successful capitalistic systems for it allows investment systems to function, and it is also critical to accurate pricing. This section not only looks at some traditional examples of the investment market, but also views other paradigms in context of the investment markets.

The wheat commodity market is a typical example of a capital market. The central idea in the wheat commodity market is that producers (i.e. farmers) and consumers (e.g bread manufacturers) protect themselves from possible catastrophes by diversifying their investments, buying futures, selling short, etc. Consequently, there is a protection from total disaster in case of unexpected drought, as they are compensated with their share of profits if the price rises unexpectedly.

An investor can invest in wheat by buying securities in the wheat market at the current price, buy futures by promising to purchase at a later time, and sell short by promising to sell a security at a later time. We may also have a merger: For example, Kansas wheat merges with Nebraska wheat to be sold through the same distributor. In addition, we can also have a split, e.g. Kansas wheat is distributed between two distributors, one for for eastern United States, and one for western United States. The spatial movements in mergers and splits, for markets like the wheat market, are proficiently modeled by our network model. In fact, our model can simulate even more sophisticated transactions. For example, a Nebraska farmer may buy some Kansas wheat, and gain from the price elevation in case of a drought in Nebraska. A Kansas farmer may buy futures in Kansas wheat to increase his profits in case of prices soaring up. A bread manufacturer (who is a consumer of the wheat) should reduce his wheat inventory if it expects a sudden price fall. On the other hand, a bread manufacturer should stock up if there is a possibility of marked increase in prices.

2.3 Viewing Multiuser Computer Systems as Investment Markets

Capital markets are not restricted to the Wall Street security markets. In fact, other paradigms can also be viewed as model for capital markets. For example, the allocation and usage of computer system resources can be viewed as a competitive market, especially when the resources are scarce and costly.For instance, the users may contend for CPU time, network availability, printer usage, disk space, cache storage, etc. Typically, several users compete with one another for these scarce resources.

In the *monetary computer resources model*, the users may be endowed with certain amount of "money" which they can use toward "purchase" of the computer resources at the current market price (determined by the current supply and demand of the resource). Money, or some similar conservative commodity, acts as a convenient intermedium for buying, selling, and trading resources. For example, if several users want the printer at simultaneously, the printer will be "sold" (i.e. assigned) to the highest bidder.

In an alternative *barter computer resources model*, the users may start with contracting some strategy to schedule these resources. Subsequently, if their schedule does not commensurate with their needs, they are allowed to trade their rights to each one of these resources for rights to the same resource at a different time, or rights to a different resource at some (same or different) time. For example, one user may trade, with another user, m hours of printer usage today for n hours of printer usage tomorrow. Similarly, a user may trade, with another user, m hours of printer usage for n bytes of disk space (where m and n are mutually agreed).

An interesting aspect of viewing the computer systems as is done in the barter computer resources model above is that money (or some other intermediate, liquid, or conservative intermedium like money)

[3] Fama argues that a premium is incorporated in stock prices to compensate the investors for the risk they take of not selecting a safer investment [8]. The premiums are tiered with respect to the associated risks. The stock market may oscillate, but it maintains a tier of premiums as a part of a larger efficient market. While the market is predictable to some extent, it may paradoxically 'predict' that you cannot beat it: An investor cannot win consistently.

is not required. This motivates us to develop a commodity model which does not involve an intermediate, liquid, or conservative quantity. Our commodity model involves investors trading commodities for other commodities, or trading an owned (or rights to own) commodities for rights to own (same or different) commodities in future. The analogy to computer systems model is a harbinger of several applications in computer systems, especially distributed systems.

We can achieve more efficient utilization of computer resources by viewing them as precious commodities as opposed to free services. Competitive schemes are generally preferable to First Come First Serve (FCFS) schemes, especially if the scheduling is congested. Creating perceived value and measured usage would make the users reluctant to intentionally waste computer resources, and they would be encouraged to use the bottleneck resources when at off-peak times, when there is less contention.

3 Two Parameter-Multiple Players

In this subsection, we extend our two-parameter stream model to accommodate several players. Subsequently, we enhance our basic model to accommodate realistic assumptions and develop practical applications. The basic two-parameter stream model with 11 farms appears in Figure 2. A farmer from any given farm i withdraws water from the streams according to its access rights, which are quantified by α_i. α_i is analogous to the investment potential of an investor in the investment market. Farmer i chooses an apportioning strategy by selecting an x_i such that $\alpha_i x_i$ fraction will be withdrawn from Stream A, and $\alpha_i(1-x_i)$ fraction will be withdrawn from Stream B. Optimal strategy selects x to maximize the amount of water acquired. The initial level of the Streams A and B is denoted by $\tilde{f}_A$ and $\tilde{f}_B$, respectively. $\tilde{f}_A^i$ and $\tilde{f}_B^i$ denotes the level of streams A and B, respectively, just before farmer i taps any water from the streams. In particular, $\tilde{f}_A^1 = \tilde{f}_A$ and $\tilde{f}_B^1 = \tilde{f}_B$.

3.1 Two Parameter-Multiple Players (Initial Level) Model

The return to the ith farmer is the amount of water it taps: $R_P^i = \alpha_i(x_i\tilde{f}_A^i + (1-x_i)\tilde{f}_B^i)$. The levels of stream for (next) farmer $i+1$ (i.e. $\tilde{f}_A^{i+1}$ and $\tilde{f}_B^{i+1}$) can be calculated from the levels of player i ($\tilde{f}_A^i$ and $\tilde{f}_B^i$) by the following formulae: $\tilde{f}_A^{i+1} = (1-\alpha_i x_i)\tilde{f}_A^i$ and $\tilde{f}_B^{i+1} = (1-\alpha_i(1-x_i))\tilde{f}_B^i$. These expressions follow from Lemma C.1.1, which is stated and proved in § C.1. Naturally, every farmer aims to optimize the expected amount of water it taps, i.e. the player i's optimal strategy would be to choose x_i so as to maximize R_P^i: $max_{x_i}[R_P^i] = max_{x_i}[\alpha_i(x_i\tilde{f}_A^i + (1-x_i)\tilde{f}_B^i)]$

3.2 Two Parameter-Multiple Players (Dynamic Adjustment) Model

In this subsection, we enhance our basic model to incorporate more realistic features and remove previous deficiencies. We introduce, formalize, and analyze a dynamic case in which the flow of the stream is continuously adjusted to successive water withdrawals, just like stock prices adjust to sizable transactions.

There are two major problems with the initial level treatment in the previous subsection. Firstly, it is unrealistic to assume that a security's price (or a stream's level) will not change during the processing of major transactions. Secondly, we get caught in repeated oscillations, instead of complete convergence to 0, when the virtual flows of the two streams are arbitrarily close to each other.

It is preferable to have a model in which the amount of water withdrawn by any farmer depends on the stream's *instantaneous* level, rather than the stream's level at the time it becomes accessible to a farmer (as assumed in the formulation in § 3.1). This is analogous to the stock market situation in which if an investor decides to sell 100,000 shares of IBM on a Monday morning, then the price of the stock is going to vary continuously during this substantial transaction. Consequently, the price of the first share sold is going to be different from the price of the last share sold (unless they are identical by a rare coincidence).

There are two reasons for variation in price during any major transaction: Firstly, any major buy or sell order is going to affect the supply-demand equilibrium, and consequently influence the price of a security. Secondly, a large buy (or sell) orders cannot be processed until there are matching sell (or buy, respectively) orders. Some transactions will have to be split into smaller trades, which will occur at different instances in time, hence the price is likely to fluctuate during the process of any large order.

The cause of repeated oscillations (as observed in third row in Table 3) is the fact that when there is only a infinitesimal difference between two streams' predictable components, the player would choose to apportion all its withdrawal rights toward tapping the stream with slightly higher level. Consequently, the absolute relative difference between two streams increases (because a quantity relatively larger than the original difference between two streams is withdrawn from the slightly larger stream). For example, if two streams had the initial level of 100 and 99, respectively, and 3% was withdrawn from the larger stream, the resultant level of the two streams will be 97 and 99, respectively. (Note: we occasionally use "level" for $\mathcal{C}$-predictable level).

The solution to our problem lies in accommodating dynamic adjustment during transactions. More precisely, the flow of the stream is continuously adjusted to reflect continuous water withdrawals, just like stock prices continually adjust during a sizable transaction order. Consequently, the levels of stream for (the next) player $i+1$ (i.e. $\tilde{f}_A^{i+1}$ and $\tilde{f}_B^{i+1}$) can be calculated from the levels of player i: $\tilde{f}_A^{i+1} = (e^{-\alpha_i x_i})\tilde{f}_A^i$ and $\tilde{f}_B^{i+1} = (e^{-\alpha_i(1-x_i)})\tilde{f}_B^i$. The return (R_P^i) in dynamic adjusting two-parameter stream model is the amount of water it taps: $R_P^i = (1-e^{-\alpha_i x_i})\tilde{f}_A^i + (1-e^{-\alpha(1-x_i)})\tilde{f}_B^i$. (This is proved in Lemma C.1.2 by solving an *exact* differential equation.) Farmer i's optimal strategy would be to choose x_i to maximize:

$$max_{x_i}[R_P^i] = max_{x_i}[(1-e^{-\alpha_i x_i})\tilde{f}_A^i + (1-e^{-\alpha(1-x_i)})\tilde{f}_B^i].$$

4 Analyzing Two-Parameter Multiplayer

4.1 Tracing Through Two-Parameter Multiplayer Examples

In a simple example at the bottom of § 1.3, we saw that one farmer was able to achieve equality between two streams by tapping from the stream with higher level. However, when the disparity between the two streams is more remarkable it may take several players before we achieve a balance between them. For example, if the initial levels of the ($\mathcal{C}(k)$-)predictable components of the streams A and B are 100 and 92, respectively, then the schedule of Table 1 would be followed.

We can see that the farmers remove any disparity between predictable components of the two streams barring small differences. These little differences are removed in the dynamic adjustment model, which also has the advantage of being more realistic. The dynamic adjustment model will follow the schedule of Table 2. These simple examples demonstrate efficiency of market, which is later exhibited by many more general cases.

When we allow short-selling, as is done in most capital markets, then it takes just one player to achieve balance in the level of the two streams. In fact, for $x = 1.8898935\cdots$ in the dynamic adjustment model, Stream A's level *falls* to $94.488618\cdots$ units and Stream B's level *rises* to $94.488618\cdots$ units in one step.

We can generalize our model to incorporate additional time-dependent streams of commodity prices. It suffices to say that each commodity, or stream (as in our model), can itself be regarded as a combination of more than one commodity, or stream, respectively. Hence, the two-parameter stream model virtually generalizes to a multiple-parameter stream model.

4.2 Main Results for Two-Parameter-Multiplayer Model

In general, a player (with some computational power $\mathcal{C}(k)$) does not know the precise level of the streams, and uses its resources and forecasting power to estimate the level. Consequently, the actual stream level can be conceptually split into two components: $\mathcal{C}(k)$-predictable component and $\mathcal{C}(k)$-unpredictable components. We consider four variations of possible characteristics of the relative nature of the $\mathcal{C}(k)$-predictable and $\mathcal{C}(k)$-unpredictable components. First three of these four models can be analyzed rigorously, and consequently we say that these models satisfy condition $C*$. The analysis developed for $C*$ models cannot be used for models that do not satisfy condition $C*$. (We leave details of these models to Appendices B.2.1 and C.2.

Four main observations for our multiplayer two-parameter stream models satisfying condition $C*$ are noted below (see Appendices C.3, C.4, D, Efor full details and proofs). Assuming all investors have the same computational power for forecasting:

Disparity between $C(k)$-predictable levels of the streams diminishes - A suave farmer (with computational power $C(k)$) will always capitalize on disparity in $C(k)$-predictable components by tapping the stream with highest $C(k)$-predictable level. This will reduce the disparity between the expected $C(k)$-predictable levels of the streams by lowering the stream(s) with higher expected $C(k)$-predictable level. This corresponds to an investor affecting the supply-demand relationship to reduce the apparent under-pricing and/or apparent over-pricing by buying apparently under-priced securities and selling apparently over-priced securities.

The affect of additional farmers - Additional farmers, each with computational power $C(k)$, will tap the stream(s) with higher expected $C(k)$-predictable components. Hence, the disparity between $C(k)$-predictable components diminishes rapidly at a near linear rate until it reaches *zero*, and subsequently it persists at *zero*. This corresponds to several investors affecting the supply-demand relationship to completely remove the apparent under-pricing and/or apparent over-pricing by buying apparently under-priced securities and selling apparently over-priced securities.

Advantage derived from $C(k)$-predictability decreases - As more and more farmers exploit $C(k)$-predictable components, the disparity between the expected $C(k)$-predictable components of the streams' flow decreases until there is no difference between the $C(k)$-predictable expected returns from various strategies available to an individual farmer. This corresponds to situation where several investors act to remove any perceived disparity between price and value of a commodity.

Scaling of the $C(k)$-unpredictable components - Even though, we assume that the $C(k)$-unpredictable components are insignificant compared to the $C(k)$-predictable components, $C(k)$-unpredictable components scale in proportion with $C(k)$-predictable components in the same stream.

The main result for the Two-Parameter-Multiplayer Stream Model is:

Theorem 4.2.1 *For our Two-parameter-Multiplayer Commodity Stream Model, the disparity between the expected $C(k)$-predictable level of two streams diminishes, and the advantage derived from $C(k)$-predictability decreases, whereas $C(k)$-unpredictable elements retain their original distribution with respect to some scaling factor.*

Theorem 4.2.1 proves efficiency in the Two-parameter-Multiplayer commodity stream model. The proof is obtained by showing that the disparity between $C(k)$-predictable levels forms a monotonically decreasing sequence bounded below by zero. (See Appendix D for details). This theorem is extended to paradigms with more than two parameters by defining a metric on maximum disparity between any pair of $C(k)$-predictable levels, and then showing that this metric defines a monotonically decreasing sequence of $C(k)$-predictable disparities bounded below by zero. (See Appendix E for details).

Theorem 4.2.2 *In our Multi-Parameter-Multiplayer Commodity Stream Model, if there is any $C(k)$-predictable disparity among the levels of two or more streams, the players would exploit it for their own benefit. Whatever initial conditions we choose, the difference between $C(k)$-predictable components of any two streams will tend to zero, with each move, until it reaches zero. Consequently, the "inefficiencies" in the market are completely removed as the $C(k)$-predictability vanishes.*

Short-selling is a tool that investors often use to gain leverage. We extend our family of models to incorporate short-selling, and show that if investors are allowed to short-sell, they would be able to extract all the predictability from our market model in a single swoop, and not leave anything for the subsequent investors to exploit.

Theorem 4.2.3 *Regardless of the initial conditions, short-selling allows one farmer to achieve what may take several farmers to achieve without short-selling: (1) Reduce the difference among the $C(k)$-predictable elements. (2) There is no disparity among the $C(k)$-predictable returns of all farmers except the first one.*

The proof is based on solving a multivariable global optimization problem with a linear constraint on the solution set. (See Appendix E.3 for details).

4.3 Varying Computing Power

We assume that all players have equivalent computational power and knowledge for our analysis in Appendices B.2.2, B.2.3, and D.3. In reality, all investors usually have access to varying amount of computational power and information. This motivates us to remove the information/computational-power equivalence from our two-parameter stream model. Without loss of generality, we assume that short-selling is allowed, because it helps in efficiently achieving equivalence of the $\mathcal{C}(k)$-predictable components, and consequently achieving efficiency.

The analysis for the model in which various investors have different information resources and computational powers leads to several interesting modulations of analysis. For example, if the order of turns is hierarchically arranged (such that players with lower computational power move first), the results are similar to the Multiplayer model satisfying condition C*.

An interesting nuance of the hierarchical model, in which a player with higher computational power ($\mathcal{C}_2(k)$) moves between two players with less computational power ($\mathcal{C}_1(k)$), leads to a fascinating analysis. Player 1's move results in equalizing its own $\mathcal{C}_1(k)$-predictable components. Player 2 equalizes its $\mathcal{C}_2(k)$-predictable components, which is a superset of the $\mathcal{C}_1(k)$-predictable components, and consequently disrupts the equivalence Player 1 had established between the $\mathcal{C}_1(k)$-predictable components. Subsequently, Player 3 is led to believe that it is extracting profit from the disparity between the $\mathcal{C}_1(k)$-predictable components.

5 Extensions: Multi-parameter and the Network Model

The basic Two Parameter-Multiplayer model can be readily generalized to Multi-parameter model by simply adding additional streams (where each additional stream corresponds to additional securities). The three-parameter stream model with 8 farmers is shown in Figure 8. The figure readily generalizes to depict more than three streams. See Appendix E for details.

We have already analyzed the effect of extracting water from flowing streams (return on investment). We add two more specialized features to our model to emulate mergers and splits, respectively, and model investment market by flow of stream in a network with three different kinds of nodes: MERGER, SPLIT, and RETURN. (Figures 7(a,b,c) show these nodes, and Figure 1 illustrates a network with these nodes in action).

The MERGER nodes are akin to corporate mergers as they combine the flow of two streams with levels f_1 and f_2 to produce a new stream with a level of f, such that $f = f_1 + f_2$. An example of merger is Kansas wheat merging with Nebraska wheat to be sold through the same distributor. The SPLIT node represents a corporation (or a product, or a market) splitting into two different corporations (or products, or markets, respectively). An example of split is Kansas wheat is being distributed between two distributors, one for for eastern United States, and one for western United States. The RETURN node denotes the return on an investor's investment or the resultant profit from allocation of resources/rights. An investor may derive a return in the wheat market by apportioning its financial resources between Kansas wheat and Nebraska wheat.

The development of our models leads to Network Model. We have already analyzed the most important node, i.e. RETURN node, in preliminary models like Two Parameter-Multiplayer Model. SPLIT and MERGER nodes give us the power to represent many more complicated markets by enhancing our model to accommodate splits and mergers without any significant additional difficulty.

A given network may be cyclic in the sense that it contains a loop in which resources can cycle forever. However, we can remove these cycles by unfolding the network over time. Such unfolding could yield up to an exponential blow up in space every time we unfold. However, it would enable us to transform any cyclic network to acyclic network. Once we have an acyclic network, we can analyze it in finite time by applying a combination the following rules:

1. Whenever we see k (two or more) streams accessible to some farm j, we equate all their apparent expected flows (assuming short-selling is allowed).
2. Whenever we see one stream splitting into k (two or more) streams, we branch the flows according to the split parameters: $p_i^{out} = \alpha_i p^{in}$, where $(\sum_i \alpha_i) = 1$
3. Whenever we see k (two or more) streams merging into one stream, we combine the flows of all the streams to compute the resultant combined flow: $\sum_i p_i^{in} = p_i^{out}$.

6 Summary of Our Main Contributions

The following are our main contributions:

1. We devise a family of very robust time-dependent commodity network models for investment markets. These models are interesting in their own right because they pose challenging combinatorial optimization problems.
2. Our family of models easily emulates various characteristics from a diverse range of markets, yet is conducive to rigorous analysis and detailed explanation. Each model is governed by a specialized blend of rules which incorporate diverse options (like selling short, buying futures, deferring investment decisions), and various features (like dynamic adjustment and price retention). Consequently, our models enable us to perform rigorous analysis and gain a better understanding of capital markets.
3. We introduce a novel scheme to represent investment markets as time-dependent flow of streams in a network. We propose a static withdrawal paradigm, which is subsequently enhanced to emulate dynamic adjustment of the resources to ever-changing scenarios. We generalize our model to allow changes over time, and to accommodate several players. We present powerful schemes to emulate mergers, splits, and returns by representing them as three different functional nodes acting on time-dependent flow of streams in a network. We describe each node in turn, and then illustrate a simple network model using these nodes. Figures 7 (a,b,c) show all different kinds of nodes, and Figure 1 illustrates a network with these nodes in action. Using this representation for security markets, we can abstract and depict all essential information by a simple network flow diagram, which can be analyzed by rules of § 5 (see § F.2 1 Appendix F.2 for detailed analysis).
4. We exploit crypto-complexity tools to model relative information, and thereby afford a very powerful paradigm of information for rigorous proofs. In particular, we analyze the capital market data as a pseudo-random sequence with respect to some crypto-complexity. This marks the first known application of crypto-complexity ideas and tools outside the realm of pure computer science.
5. Our major contributions, besides proposing new models, are rigorous proofs for relating efficiency to predictability in our models of investment markets. Using our models, we determine circumstances where we can rigorously prove that the prices of securities accurately reflect their true values in the investment markets, and prove efficiency by showing that any predictability in stream levels will be exploited by the players so as not to leave any "inefficiency" in the levels.[4] Subsequently, we can identify the situations indicating efficiency in capital markets - which are complicated to analyze directly.
6. We incorporate short-selling into our model: Short-selling is a tool that investors often use in capital markets to protect themselves from substantial losses if the price of a security falls. We show that if investors are allowed to short-sell, they would be able to extract all the predictability from our market model in a single swoop, and not leave anything for the subsequent investors to exploit. Consequently, it takes only one player to remove any "inefficiency" (or predictability) as the first player's portfolio can exploit all inefficiency for the player's own benefit, and not leave any exploitable "inefficiency" (as prices will efficiently and accurately adjust to reflect any new information).
7. Our analysis can be applied to analyze other markets. Furthermore, the apparent relationship with distributed systems to the *barter computer resources model* promises applications in that area.

In simple terms, our thesis is that: "You snooze, you lose" - No matter what profitable information or valuable idea you have, implement it immediately, because if someone else implements it before you do, there would not be any advantage left for you to exploit.

[4]Tables 1 through 4 sketch the trace followed by predictable components of a two-parameter stream model. The analysis readily generalizes to multi-parameter stream model. Tables 5 and 6 offer a sample trace of predictable components in a three-parameter stream model.

References

[1] Louis Bachelier. *Théorie de la Speculation.* Gauthier-Villars, Paris, France, 1900.

[2] Fischer Black. Capital market equilibrium with restricted borrowing. *Journal of Business*, pages 444–454, July 1972.

[3] Fischer Black, Michael Jensen, and Myron Scholes. The capital assets pricing model: some empirical tests. In *Jenson (ed.)*, pages 79–121, 1972.

[4] Manuel Blum and Silvio Micali. How to generate cryptographically strong sequences of pseudo random bits. In *FOCS 23*, volume 13, pages 112–117, 1982.

[5] Manuel Blum and Silvio Micali. How to generate cryptographically strong sequences of pseudo random bits. *SIAM Journal of Computing*, 13:850–864, 1984.

[6] Michael J. Brennan. Capital market equilibrium with divergent borrowing and lending rates. *Journal of Financial and Quantitative Analysis*, pages 1197–1205, December 1971.

[7] Eugene F. Fama. Efficient capital markets: A review of theory and empirical work. *Journal of Finance*, 25:383–417, May 1970.

[8] Eugene F. Fama. *Foundations of Finance.* Basic Books, Inc., New York, New York, 1976.

[9] Eugene F. Fama and James D. Macbeth. Risk, return, and equilibrium: Empirical tests. *Journal of Political Economy*, pages 607–636, May-June 1971.

[10] M.G. Kendall and A.S. Stuart. *The Advanced Theory of Statistics, volume 3, Design & Analysis & Time Series.* Hafner, New York, New York, 2nd edition, 1968.

[11] John Lintner. The valuation of risk assets and the selection of risky investment in stock portfolios and capital budgets. *Review of Economics and Statistics*, pages 13–37, February 1965.

[12] Harry Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, March 1952.

[13] Harry Markowitz. *Portfolio Selection: Efficient Diversification of Investments.* John Wiley, New York, New York, 1959.

[14] Robert C. Merton. An analytical derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis*, September 1972.

[15] Merton Miller and Myron Scholes. Rates of return in relation to risk: A re-examination of some recent findings. In *Jenson (ed.)*, pages 47–78, 1972.

[16] Adi Shamir. A polynomial time algorithm for breaking the basic Merkle-Hellman cryptosystem. In *FOCS 23*, pages 145–152, 1982.

[17] Adi Shamir. A polynomial time algorithm for breaking the basic Merkle-Hellman cryptosystem. *IEEE Transactions on Information Theory*, IT-30:699–704, 1984.

[18] William F. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, pages 425–442, September 1964.

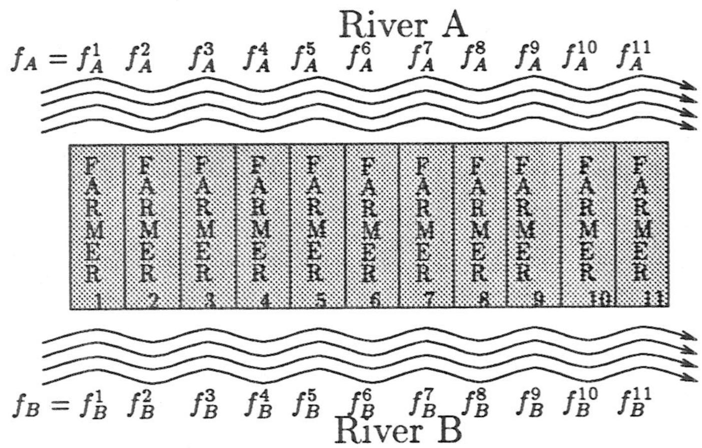

Figure 2: The Basic Two-Parameter Stream Model

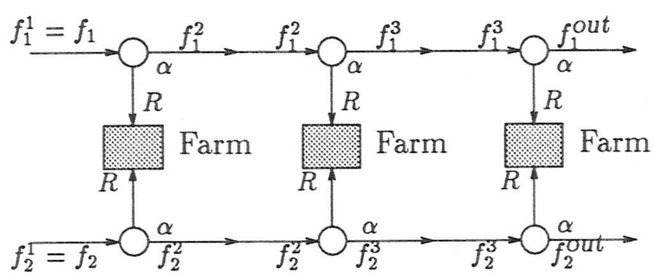

Figure 3: Network Illustration for Two-parameter Stream Model

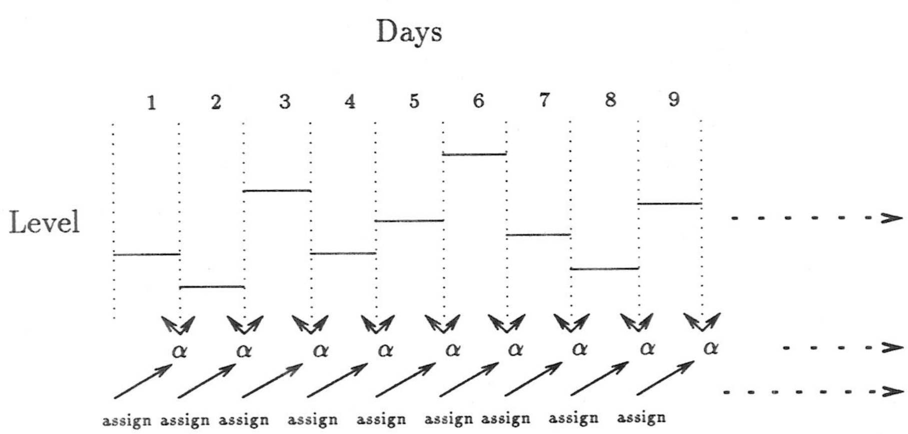

Figure 4: The Basic One-Parameter Stream Model

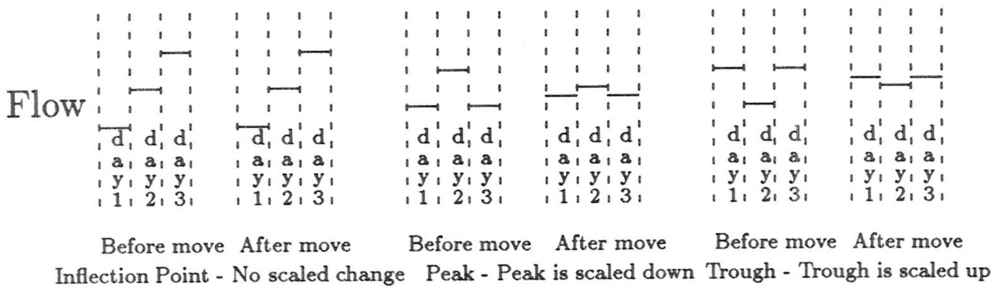

Figure 5: How a Player's Move Affects Inflection Point, Peak, and Trough

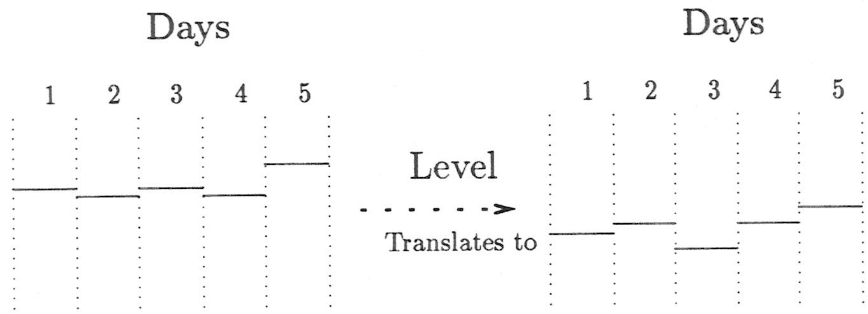

Figure 6: The Oscillations Near Convergence

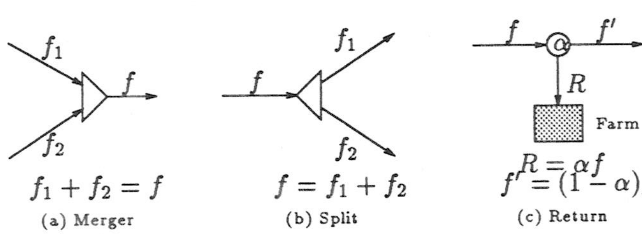

Figure 7: Network Nodes Illustrated

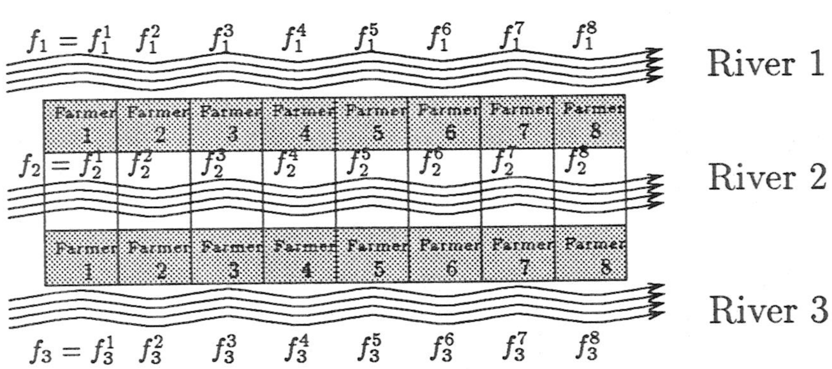

Figure 8: The Three-Parameter Stream Model

Appendices to:
"Crypto-complexity Based Models of Efficiency in Capital Markets"

by *Salman Azhar* and *John H. Reif*

For the convenience of the referees, we have included a set of appendices with detailed proofs of all the theorems.

A Overview of the Appendices

§ B introduces several original models leading to the network model which emulates investment markets by time-dependent flow in a network. The flow across a given edge over time forms a pseudo-random time sequence with respect to various cryptographic complexities, and investors face a choice of allocating and scheduling their limited resources between accessible investments, so as to maximize their returns with respect to the information available to them. These allocation and scheduling emulates decisions to buy and sell as well as more sophisticated notion of options market and short selling. This is the first known extension of crypto-complexity theory to time-dependent networks.

§ C concentrates on One-period Two-parameter models, and uses the analogy of "A Village with Two Streams" to develop intuition and to gain expositional ease. § D extends the Two-parameter model to accommodate several players, who take turns to move, in some arbitrary order, leading to Multiple-Player Two-parameter Model.

§ E serves as a prelude to the Network Models of § F by defining Multiparameter models. § E develops the considerable formal machinery that lends a fresh perspective to the Two-Parameter Models, and provides the foundation for the Network Models introduced in § F. Several tools of § E will prove to be useful for future research in this area.

§ E describes situations in our stream models for which we prove efficiency by showing that all apparent components of the stream level will be exploited by the players so as not to leave any predictability in the stream level. In our model, investors face a choice of allocating their resources between accessible investments, so as to maximize their returns with respect to the information available to them. If short-selling is allowed, we show that the prices will adjust even more efficiently and accurately to reflect any new information. If short-selling is disallowed, then the first few (finite number) of investors can take advantage of these inefficiencies.

§ F describes a network of time-dependent stream flow to emulate capital markets. Our network model is based on time-dependent stream flow in a network with functional nodes, which regulate the flow according to the specified rules. There are three main nodes RETURN, MERGER, and SPLIT, and each has a different function.

We conclude that no matter what profitable information or valuable idea leads to exploitable inefficiency, there would not be any inefficiency left for one to exploit, unless it is exploited immediately, because otherwise others will squeeze all profits Possible from the inefficiency. In layman's terms, our thesis is that: "You snooze, you lose".

B Our Models for Commodity Markets

This section presents an overview of the wide variety of models introduced in this paper. We highlight the similarities and differences in the models and present formal and rigorous results as theorems.

B.1 One-Period Two-Parameter Models: A Village with Two Streams

Our stream model is analogous to the commodity market, where the investors buy and sell real items. The model can also emulate the stock and bond market, but this correspondence is obvious only after quantitative formalization of returns is developed. In fact, we shall observe that the only formal difference between our stream commodity market model and the stock market model is that the formula employed for calculating returns is different.

B.1.1 Our One-Period Two-Parameter Stream (Initial Level) Model

In our stream model of investment markets, withdrawing water from a stream is analogous to return on investment: The return to a farmer from a stream is the amount of water it extracts from that stream.

Imagine a farm adjacent to a stream X. Let $\tilde{f}_X$ be the flow apparent to a farmer on the farm. If the farmer can withdraw a fraction γ of the flow apparent to it, then its return $(\tilde{R}_X)$ is given by $\tilde{R}_X = \gamma \tilde{f}_X$, because a fraction γ is removed from the stream X with initial flow $\tilde{f}_X$ virtual to the farmer. The outgoing flow $(\tilde{f}_X^{out})$ will be $\tilde{f}_X - \tilde{R}_X$ or $\tilde{f}_X^{out} = (1-\gamma)\tilde{f}_X$.

We will first look at simplest of these cases with one player situated adjacent to two streams A and B with levels $\tilde{f}_A$ and $\tilde{f}_B$. $\tilde{f}_A = p_A + \tilde{q}_A$, and $\tilde{f}_B = p_B + \tilde{q}_B$, where the player has computational power $\mathcal{C}(k)$, and p_A and p_B are $\mathcal{C}(k)$-predictable, while $\tilde{q}_A$ and $\tilde{q}_B$ are $\mathcal{C}(k)$-unpredictable random variables.

Each farmer has rights to withdraw a fraction α of water from the stream it chooses to tap, where α corresponds to the investment resources in capital markets. The farmer has to choose an apportioning strategy by selecting an x such that αx fraction will be withdrawn from Stream A, and an $\alpha(1-x)$ fraction will be withdrawn from Stream B. Optimal strategy selects x to maximize the amount of water acquired. We disallow short-selling (in the initial level models only) by requiring $0 \leq x \leq 1$.

The player's return (R_P) on some portfolio P (i.e. apportionment P) is given by the following function: $R_P = \alpha(x\tilde{f}_A + (1-x)\tilde{f}_B) = \alpha(xp_A + (1-x)p_B + x\tilde{q}_A + (1-x)\tilde{q}_B)$. Consequently, $E[R_P] = \alpha(xE[\tilde{f}_A] + (1-x)E[\tilde{f}_B]) \;\; = \alpha(xp_A + (1-x)p_B + xE[\tilde{q}_A] + (1-x)E[\tilde{q}_B])$.

B.1.2 Our One-Period Two-Parameter Stream (Dynamic Adjustment) Model

In reality, the amount of water withdrawn by any farmer depends on the instantaneous level of the stream (due to the continuously changing in level during a transaction), rather than the stream's initial level when it becomes accessible to the farmer (as assumed in the formulation above). This is analogous to the stock market situation in which if an investor decides to sell 100,000 shares of IBM on a Monday morning, the price of the stock is going to vary continually while this big volume is being sold. Consequently, the price of the first share sold is going to be different from the price of the last share sold (unless they are identical by a rare coincidence). We introduce, formalize, and analyze a dynamic case in which the flow of the stream is continuously adjusted to successive water withdrawals, just like stock prices continually adjust to a sizable transaction.[5]

When we incorporate dynamic adjustment, the return from stream X to a farmer of the adjacent farm is given by $\tilde{R}_X = (1-e^{-\gamma})\tilde{f}_X$. The outgoing flow $(\tilde{f}_X^{out})$ will be $\tilde{f}_X - \tilde{R}_X$, and hence $\tilde{f}_X^{out} = e^{-\gamma}\tilde{f}_X$.

Now, let us turn to the case of two streams A and B, with levels $\tilde{f}_A$ and $\tilde{f}_B$. $\tilde{f}_A = p_A + \tilde{q}_A$, and $\tilde{f}_B = p_B + \tilde{q}_B$, where the player has computational power $\mathcal{C}(k)$, and p_A and p_B are $\mathcal{C}(k)$-predictable, while $\tilde{q}_A$ and $\tilde{q}_B$ are $\mathcal{C}(k)$-unpredictable random variables.

Each farmer has rights to withdraw a fraction α of water from the stream it chooses to tap, where α corresponds to the investment resources in capital markets. The farmer has to choose an apportioning strategy by selecting an x such that an αx fraction will be withdrawn from Stream A, and an $\alpha(1-x)$ fraction will be withdrawn from Stream B. Optimal strategy selects x to maximize the amount of water acquired. x can be any real number unless we disallow short-selling, and in that case $0 \leq x \leq 1$. In the dynamic case, the player withdraws from *instantaneous* flow of the streams (which is dynamically adjusting to reflect any water extracted). Consequently, the player's return on some portfolio P (i.e. allocation P) is given by the following function: $R_P = (1-e^{-\alpha x})\tilde{f}_A + (1-e^{-\alpha(1-x)})\tilde{f}_B = (1-e^{-\alpha x})(p_A + \tilde{q}_A) + (1-e^{-\alpha(1-x)})(p_B + \tilde{q}_B)$. Consequently, $E[R_P] = (1-e^{-\alpha x})(p_A + E[\tilde{q}_A]) + (1-e^{-\alpha(1-x)})(p_B + E[\tilde{q}_B])$.

Once we have developed this basic formulation, we are interested in the optimal strategy pursued by the player. We shall commence by analyzing expected return for risk apathetic players, and reserve risk sensitive return analysis for later. The formulae used in risk sensitive analysis for standard deviations of returns in our two-parameter one-period stream model are stated in Appendix G.

B.1.3 Classical One-Period Two-Parameter Models

Classically, the return on stock i at time t is given by the following expression: $\tilde{R}_{it} = \frac{\tilde{f}_{it} - f_{i,t-1}}{f_{i,t-1}}$, where $\tilde{R}_{it}$ is the random variable denoting the return on stock i at time t, $\tilde{f}_{it}$ is the price of stock i at time t (which incorporates all dividends, mergers, and splits), and $f_{i,t-1}$ is the price of the stock i at time $t-1$. Following the standard notation of finance, we use the $\tilde{}$ notation above a variable to indicate that it is a random variable.

The classical one-period, two-parameter model is due to Markowitz [12, 13]. At time $t=1$, the investor has wealth w_1 which he must allocate to consumption c_1 and investment $(w_1 - c_1)$ in some portfolio of securities. The consumption at time $t=2$ is $\tilde{c}_2 = (w_1 - c_1)\tilde{R}_P$, where $\tilde{R}_P$ is the return on portfolio P computed by $\tilde{R}_P = x\tilde{R}_q + (1-x)\tilde{R}_s$. Here, $\tilde{R}_q$ and $\tilde{R}_s$ are the returns on available investments q and s, whereas x determines the apportionment of investment between q and s. We model the uninvested wealth w_1 as wealth invested in an investment that retains its value, without any profit or loss. The central issues in classical two-parameter model are expected return, risk potential, market equilibrium, and correlation between $\tilde{R}_q$ and $\tilde{R}_s$. The total utility for the investor is $w_1 + w_2$.[6]

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[5] Our model's *continuous* adjustment is slightly different from the *continual* adjustment that occurs in actual investment markets, but our *continuous* adjustment assumption facilitates the modeling without any significant deviation.

[6] We refer the reader to Chapters 7 through 9 of Fama's book on security markets [8] for a concise exposition of issues related to the two-parameter model. For more detailed discussions, the reader is referred to the works of Markowitz [12, 13],

Farmer (i)	Stream A p_A		q_A	Stream B p_B		q_B	$\lvert\frac{p^i_A - p^i_B}{(1-\alpha/2)^{(i-1)}}\rvert$	x_i	$\frac{R^i_{P(x_i)}}{(1-\alpha/2)^{i-1}}$	$\frac{R^i_{worst}}{(1-\alpha/2)^{i-1}}$	Difference
1	100	+	Q	92	+	Q	8.00	1	$3.00 + 0.03Q$	$2.76 + 0.03Q$	0.24
2	97	+	$0.97Q$	92	+	Q	5.08	1	$2.95 + 0.03Q$	$2.80 + 0.03Q$	$0.15 - 0.00Q$
3	94.09	+	$0.94Q$	92	+	Q	2.15	1	$2.91 + 0.03Q$	$2.84 + 0.03Q$	$0.07 - 0.00Q$
4	91.26	+	$0.91Q$	92	+	Q	0.77	0	$2.89 + 0.03Q$	$2.86 + 0.03Q$	$0.03 - 0.00Q$
5	91.26	+	$0.91Q$	89.24	+	$0.97Q$	2.15	1	$2.91 + 0.03Q$	$2.84 + 0.03Q$	$0.07 - 0.00Q$
6	88.52	+	$0.88Q$	89.24	+	$0.97Q$	0.77	0	$2.89 + 0.03Q$	$2.86 + 0.03Q$	$0.03 - 0.00Q$
7	......		......	......		......	...	...	...	...	...
8	......		......	......		......	...	...	...	...	...

Table 3: Changes in Levels for Two-Parameter Stream (Initial Level) Model

B.2 Two Parameter-Multiplayer Stream Model

The Two Parameter-Two Player Stream Model can be readily generalized to Two Parameter-Multiple Player Stream Model. The return to the ith farmer is the amount of water it taps. For Initial Level Model: $R^i_P = \alpha_i(x_i\tilde{f}^i_A + (1-x_i)\tilde{f}^i_B)$; $\tilde{f}^{i+1}_A = (1-\alpha_i x_i)\tilde{f}^i_A$ and $\tilde{f}^{i+1}_B = (1-\alpha_i(1-x_i))\tilde{f}^i_B$. Farmer i's optimal strategy would be to choose x_i so as to maximize R^i_P: $max_{x_i}[R^i_P] = max_{x_i}[\alpha_i(x_i\tilde{f}^i_A + (1-x_i)\tilde{f}^i_B)]$. For the Dynamic Adjustment Model: $R^i_P = (1-e^{-\alpha_i x_i})\tilde{f}^i_A + (1-e^{-\alpha(1-x_i)})\tilde{f}^i_B$; $\tilde{f}^{i+1}_A = (e^{-\alpha_i x_i})\tilde{f}^i_A$ and $\tilde{f}^{i+1}_B = (e^{-\alpha_i(1-x_i)})\tilde{f}^i_B$. Farmer i's optimal strategy would be to choose x_i so as to maximize R^i_P: $max_{x_i}[R^i_P] = max_{x_i}[(1-e^{-\alpha_i x_i})\tilde{f}^i_A + (1-e^{-\alpha(1-x_i)})\tilde{f}^i_B]$.

B.2.1 Information Theoretic Variations of Two Parameter-Multiple Players Model

In general, player i does not know $\tilde{f}^i_A$ and $\tilde{f}^i_B$ precisely. Using earlier notation, we denote $\tilde{f}^i_A = p^i_A + \tilde{q}^i_A$, and $\tilde{f}^i_B = p^i_B + \tilde{q}^i_B$, such that p^i_A and p^i_B are $C_i(k)$-predictable components, $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are $C_i(k)$-unpredictable components, where $C_i(k)$ is the exact amount of computational power at farmer i's disposal.

We consider four possible characteristics of the unpredictable components ($\tilde{q}$):

M_1 *The $C_i(k)$-unpredictable elements are negligibly small relative to the $C_i(k)$-predictable elements, i.e. $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are negligibly small compared to p^i_A and p^i_B, respectively. Furthermore, both $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are negligibly small compared to $|p^i_A - p^i_B|$.* In this case, the ith farmer's optimal strategy is dependent only on the values of p^i_A and p^i_B.

M_2 *The $C_i(k)$-unpredictable elements are known constants, i.e. $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are known constants Q^i_A and Q^i_B.* In this case, the ith farmer's optimal strategy depends on Q^i_A, Q^i_B, p^i_A and p^i_B.

M_3 *The expectation of $C_i(k)$-unpredictable elements are known constants, even though the actual values are not known, i.e. the expected values of $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are known constants, $E[\tilde{q}^i_A] = Q^i_A$ and $E[\tilde{q}^i_B] = Q^i_B$.* In this case, the ith farmer's optimal strategy depends on Q^i_A, Q^i_B, p^i_A and p^i_B.

M_4 *There are no restrictions on the $C_i(k)$-unpredictable elements, except that their upper and lower bounds are known.*[7] In this case, the ith farmer's optimal strategy depends on four variables: $\tilde{q}^i_A$, $\tilde{q}^i_B$, p^i_A and p^i_B.

Each of the above models (M_1, M_2, M_3, and M_4) are subclassified with respect to Initial Level Model (ILM_1, ILM_2, ILM_3, and ILM_4, respectively) and Dynamic Adjustment Model (DAM_1, DAM_2, DAM_3, and DAM_4, respectively). We shall analyze ILM_1 and DAM_1 to build our intuition before we delve into formal details.

B.2.2 A Brief Analysis of ILM_1

In accordance with the ILM_1 paradigm, we assume that the $C_i(k)$-unpredictable components of the flow are insignificant compared to respective predictable components, i.e. $\tilde{q}^i_A << p^i_A$, $\tilde{q}^i_B << p^i_B$, $\tilde{q}^i_A << |p^i_A - p^i_B|$, and $\tilde{q}^i_B << |p^i_A - p^i_B|$. We further assume that all farmers have exactly the same computational resources, i.e. $C_i(k) = C(k)$ for all i.

Suppose, initially $\tilde{f}_A = \tilde{f}^1_A = 100 + \tilde{q}^1_A$, and $\tilde{f}_B = \tilde{f}^1_B = 92 + \tilde{q}^1_B$. The optimal strategy will be to maximize $max_{x_i}[x_i 100 + (1-x_i)92]$. If $\alpha_i = 3\%$ for all farmers i, then the schedule of Table 3 would be followed.

The disparity between the scaled $C(k)$-predictable components of the two streams diminishes at least every alternate term[8]. Observe that when as the disparity between predictability approaches zero, alternate terms $\lvert\frac{p^i_A - p^i_B}{(1-\alpha/2)^{(i-1)}}\rvert$ starts oscillating (initially between 2.15 and 0.77 approximately), and the oscillations becomes

Merton [14], Sharpe [18], Lintner [11], Brennan [6], and Black [2]. Empirical test of the two-parameter model have also been performed by Fama and Macbeth [9], Miller and Scholes [15], and Black et al. [3].

[7] $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are bounded but are not necessarily negligibly small compared to p^i_A and p^i_B, respectively.

[8] $\langle 1, 2, 3, 4, 6, 8, 10, ...\rangle$ in some cases, and $\langle 1, 2, 3, 4, 5, 7, 9, 11, ...\rangle$ in some other cases.

Farmer	Stream A			Stream B			$\lvert\frac{p_A^i - p_B^i}{(1-\alpha/2)^{(i-1)}}\rvert$	x_i	$\frac{R^i_{P(x_i)}}{(1-\alpha/2)^{i-1}}$	$\frac{R^i_{worst}}{(1-\alpha/2)^{i-1}}$	Difference
(i)	p_A		q_A	p_B		q_B					
1	100	+	$\tilde{Q}$	92	+	$\tilde{Q}$	8.00	1.0	$2.96 + 0.03\tilde{Q}$	$2.72 + 0.03\tilde{Q}$	0.24
2	97.04	+	$0.970\tilde{Q}$	92	+	$\tilde{Q}$	5.11	1.0	$2.91 + 0.03\tilde{Q}$	$2.76 + 0.03\tilde{Q}$	$0.15 - 0.00\tilde{Q}$
3	94.18	+	$0.941\tilde{Q}$	92	+	$\tilde{Q}$	2.15	0.89	$2.87 + 0.03\tilde{Q}$	$2.80 + 0.03\tilde{Q}$	$0.07 - 0.00\tilde{Q}$
4	91.70	+	$0.917\tilde{Q}$	91.70	+	$0.997\tilde{Q}$	0.00	$\frac{1}{2}$	$2.85 + 0.03\tilde{Q}$	$2.85 + 0.03\tilde{Q}$	$0 + 0.00\tilde{Q}$
5	90.33	+	$0.903\tilde{Q}$	90.33	+	$0.982\tilde{Q}$	0.00	$\frac{1}{2}$	$2.85 + 0.03\tilde{Q}$	$2.85 + 0.03\tilde{Q}$	$0 + 0.00\tilde{Q}$
6	88.99	+	$0.890\tilde{Q}$	88.99	+	$0.967\tilde{Q}$	0.00	$\frac{1}{2}$	$2.85 + 0.03\tilde{Q}$	$2.85 + 0.03\tilde{Q}$	$0 + 0.00\tilde{Q}$
7	...		...	...		...	...	...	...	...	...
8	...		...	...		...	...	...	...	...	...

Table 4: Changes in Levels for Two-Parameter Stream (Dynamic Adjustment) Model

successively smaller. Similarly, the difference between the optimal and the worst return starts oscillating (initially commmencing between $0.07 - 0.00\tilde{Q}$ and $0.03 - 0.00\tilde{Q}$ approximately), and these oscillations become successively smaller.

Our results can be summarized as follows by observing two trends in the following theorem:

Theorem B.2.1 *If all the players pursue optimal strategy then: $|p_A^i - p_B^i|$ decreases monotonically and approaches zero. Furthermore, the advantage derived from pursuing optimal strategy diminishes as the disparity between the predictable components approaches zero.*

ILM_2 and ILM_3 can be analyzed by substituting $p_A^i + Q_A^i$ for p_A^i, and substituting $p_B^i + Q_B^i$ for p_B^i. Note that the analysis of ILM_1, ILM_2, and ILM_3 is similar, and we say that these models satisfy condition C*. One the other hand, ILM_4 is analyzed later in § D.5 with a different and more intricate analysis, and we say that ILM_4 does not satisfy condition C* because the techniques of C* models cannot be used for ILM_4.

B.2.3 A Brief Analysis of DAM_1

The analysis of DAM_1 is similar to the analysis of ILM_1 except the difference in the formula for computing returns leads to more favorable results. In accordance with the assumptions of DAM_1, we assume $\tilde{q}_A^i << p_A^i$, $\tilde{q}_B^i << p_B^i$, $\tilde{q}_A^i << |p_A^i - p_B^i|$, and $\tilde{q}_B^i << |p_A^i - p_B^i|$. We further assume that all farmers have exactly the same computational resources, i.e. $\mathcal{C}_i(k) = \mathcal{C}(k)$ for all i.

Suppose, initially $\tilde{f}_A = \tilde{f}_A^1 = 100 + \tilde{q}_A^1$, and $\tilde{f}_B = \tilde{f}_B^1 = 92 + \tilde{q}_B^1$. The optimal strategy will be to maximize $max_{x_i}[(1 - e^{\alpha x_i})100 + (1 - e^{\alpha(1-x_i)})92]$. If $\alpha_i = 3\%$ for all farmers i, then the schedule of Table 4 would be followed.

When we start out with $p_A^1 = 100$ and $p_B^1 = 92$, x_1 and x_2 are chosen to be 1.00 for the first two rounds to get the maximum benefit from Stream A. In the third round, we need to minimize $100 - 100e^{-0.06-0.03x_3} + 92 - 92e^{-0.03(1-x_3)}$. Differentiating, equating to zero, and then simplifying yields, $x_3 = -\frac{0.03+\ln(0.92)}{0.06} = 0.88969348$. This forces equality of p_A^i and p_B^i. Once equality has been achieved the ith farmer's optimal strategy will be to choose $x_i = 0.5$, and this choice of x_i maintains equality in all subsequent terms. The result is summarized in the following theorem:

Theorem B.2.2 *If all the players pursue optimal strategy then: $|p_A^i - p_B^i|$ decreases monotonically until it reaches zero. Furthermore, the advantage derived from pursuing optimal strategy diminishes as the disparity between the predictable components approaches zero, until all strategies are equivalent.*

We observe that from Table 4 that $|\frac{p_A^i - p_B^i}{(1-\alpha/2)^{(i-1)}}|$ diminishes rapidly until it reaches *zero*. However, it remains to be gauged how fast it approaches zero, i.e. find $N(\epsilon)$ such that $|\frac{p_A^1 - p_B^1}{(1-\alpha/2)^{(i-1)}}| < \epsilon$ for $i > N(\epsilon)$. Without loss of generality, assume $p_A^1 > p_B^1$. Then $e^{-\alpha n} p_A^1 < p_B^1 < e^{-\alpha(n+1)} p_A^1$ implies that $p_A^i = p_B^i$ for $i > n$. Dividing through by p_B^1 and taking natural logarithms, we can conclude:

Corollary B.2.1 *$p_B^{n+1} = p_A^{n+1}$ for the first time if and only if $n \geq \lceil \frac{\log(p_A^1/p_B^1)}{\alpha} \rceil$.*

DAM_2 and DAM_3 satisfy condition C*, and can be analyzed by substituting $p_A^i + Q_A^i$ for p_A^i, and substituting $p_B^i + Q_B^i$ for p_B^i. DAM_4 does not satisfy condition C* because the techniques of C* models cannot be used for its analysis.

B.2.4 Two-parameter-Multiplayer Player (with Short-Selling)

Short-selling is a tool employed by investors to increase their leverage by allowing one investor with limited funds to be more influential in the market than it could have been otherwise. Short-selling involves an investor borrowing some securities from another investor and selling them in the market. He is said to have a "short" position in those shares. On the other hand, owning a security is termed as "long" position. In terms of our stream-parameter model, a player may go "short" by exchanging his rights to withdraw water from one stream for rights to withdraw from another stream.

The quantitative effect of short-selling, is that the value of x is allowed to be more than 1 or less than 0. Thus, if $x < 0$ then Stream A is sold short, and if $x > 1$ then Stream B is sold short. Short-selling does not integrate with ILM (Initial Level Models) since a player can extract unlimited return because the levels (prices) are insensitive to large transactions. In the running example, if short-selling is allowed, it takes only one farmer to act before we reach the steady state of $|p_A^i - p_B^i| = 0$, and for $\alpha = 0.03$, optimal value of $x = 1.8896935$, which gives $p_A^2 = p_B^2 = 94.488618$.

Theorem B.2.3 *Regardless of the initial conditions, if condition C^* is satisfied then short-selling allows* <u>*one*</u> *farmer to achieve what takes several farmers to achieve:*

1. *Reduce the difference between the $C(k)$-predictable elements to zero.*
2. *There is no disparity between the $C(k)$-predictable return of various strategies for all farmers except the first one, unless some farmer deviates from the optimal strategy.*

B.3 Multiple-Period One-Parameter Model

For our stream model, we have already described one-period two-parameter paradigm with one player as well as several players. Now, we extend our model to allow players to optimally invest over time (modeled by variations of the stream levels over time), and extend the scheduling strategy to be optimal over a period of time. We will initiate the discussion by a simple case of one-parameter stream model in which the players can defer their rights by one day if they so desire. Subsequently, we will extend our model with more powerful enhancements.

B.3.1 Multiple-Period One-Parameter Model with One-Period Limited Deferment

Consider the scenario with several farms banking on one stream. Let the flow of the stream be $\tilde{f}_t = p_t + \tilde{q}_t$, where p_t is $C(k)$-predictable by all players and $\tilde{q}_t$ is an $C(k)$-unpredictable random variable. For the purpose of strategy formulation, we assume that only the $C(k)$-predictable component is relevant, since the $C(k)$-unpredictable component cannot influence the strategy. Every farmer has to formulate a strategy to draw water on a given day, or defer that day's rights to the next day (but can't defer to the day after next). The strategy formulation can be perceived as the farmer allocating the schedule at midnight for the following midnight as illustrated in Figure 4. On ith day, the farmer must decide to schedule the water pump between ith and $i+1$st day.

One fascinating observation is that this multiple-period one-parameter stream model with one-period limited deferment is analogous to the one-period two-parameter stream model. The level of the stream on adjacent days in the one-parameter stream model is virtually identical as the two streams on the same day in the two-parameter stream one-period model: Instead of allocating the water pump between two streams the farmer must allocate the water pump between two (adjacent) days. Nevertheless, we perform a slightly different analysis on our multiple-period model to lend some intuition and variety of perspective.

For simplicity of notation, we scale the level by $1 - \alpha$ every time we pass a farm, regardless of what fraction the farmer has withdrawn (between 0 and 2α).

There are three possible cases that the flow of a stream can find itself depending on the relative flows on adjacent days. These are illustrated in Figure 5, and explained below.

Inflection Point : *Higher flow on one side, Lower flow on the other side.* $\tilde{f}^{i+1} = (1-\alpha)\tilde{f}^i$. There is no scaled change, i.e $\tilde{f}^{i+1}_{scaled} = \tilde{f}^i_{scaled}$.

Peak : *Lower flow on both sides.* $\tilde{f}^{i+1} = (1-\alpha)(1-\alpha)\tilde{f}^i$. There is a scaled change of $(1-\alpha)$, and $\tilde{f}^{i+1}_{scaled} = (1-\alpha)\tilde{f}^i_{scaled}$.

Trough : *Higher flow on both sides.* $\tilde{f}^{i+1} = \tilde{f}^i$. There is no change in magnitude, but there is a scaled change of $\frac{1}{(1-\alpha)}$, and $\tilde{f}^{i+1}_{scaled} = \frac{1}{(1-\alpha)}\tilde{f}^i_{scaled}$.

The results of these cases is summarized in the lemma below:

Lemma B.3.1 *After scaling, all local maxima are reduced by a factor of $(1-\alpha)$, all local minima are increased by a factor of $\frac{1}{(1-\alpha)}$, whereas inflection nodes exhibit no scaled change.*

In simple cases, we can easily see this scaling would converge the amplitude to zero. Unfortunately, there can be a potential problem in our aggressive statement. It is possible that the scaled difference increases after a player extracts its maximum benefit. This happens when the predictable components of the two streams are very close together. For example, if $\alpha = 5\%$, and the sequence of values on five consecutive days is $\langle ..., 98, 99, 100, 101, 102, ... \rangle$, then after the first iteration , the middle three terms will be $\langle ..., 94.05, 90.25, 95.95, ... \rangle$ which when scaled by $\frac{1}{0.95}$ to give $\langle ..., 99, 95.00, 101, ... \rangle$. Evidently, the maximum relative difference between two consecutive scaled values has increased after the move according to the rules. Consequently, we should be extremely careful before we draw any conclusions about the convergence of maximum difference. This problem is illustrated in Figure 6 (which magnifies the effect for visual aid).

The worst case example is that when the global maximum is infinitesimally close to global minimum, e.g maximum is $f^i = f(1+\epsilon)$ and the minimum is $f^j = f$, for some infinitesimal $\epsilon > 0$. In such a case, action of one farmer will decrease the scaled f^i to $(1-\alpha)f(1+\epsilon)$, increase the scaled $f^j = \frac{1}{1-\alpha}f$. Consequently, the relative difference will increase from $1+\epsilon$ to $\frac{1}{(1-\alpha)^2(1+\epsilon)}$.

B.4 A Network Representation

In this subsection, we add specialized features to our model to emulate mergers, splits, and returns. We do this by representing the flow of the stream by a network with three different kinds of nodes. We describe each node in turn, and then explain a simple network model using these nodes. Figures 7 (a,b,c) show all different kinds of nodes.

The MERGER nodes are akin to corporate mergers as they combine the flow of two streams with levels f_1 and f_2 to produce a new stream with a level of f, such that $f = f_1 + f_2$. The SPLIT node represents a corporation (or a product, or a market) splitting into two different corporations (or products, or markets, respectively). The RETURN node denotes the return on an investor's investment or the resultant profit from allocation of resources/rights.

The development of our models has been leading to the network model. We have already analyzed the most crucial node, i.e. RETURN node, in previous models. SPLIT and MERGER nodes give us the power to represent many more complicated markets by enhancing our model to accommodate splits, mergers, and dividends, as described in the previous subsection. Once, we have chosen to represent a security market by the representation above, we can abstract all information about it and depict it by a simple network flow diagram. The analysis of this flow diagram is much simpler than we might expect: In fact, this simplicity can be attributed to the rigorous tools developed earlier.

A given network may be cyclic in the sense that it contains a loop in which resources can cycle forever. However, we can remove these cycles by unfolding the network over time. Such unfolding would yield a blow up to an exponential blow up in space every time we unfold. However, it would enable us to analyze any cyclic network within a finite amount of time.

Once we are presented with a acyclic network, it is a matter of applying three rules to complete the analysis. These rules are listed below:

Clamp Rule : Whenever we see k (two or more) streams accessible to some farm j, we equate all their apparent expected flows according to Equation 13 (which assumes short-selling is permitted): $p_i^{out} = e^{-\frac{1}{k}\alpha}(\prod_{i=1}^{k} p_i^{in})^{1/k}$. It is apparent that Clamp rule has an effect similar to voltage clamps in electrical networks: It forces all associated stream flows to be identical. Once that happens, there is no predictability left for subsequent investors to exploit.

Split Rule : Whenever we see one stream splitting into k (two or more) streams, we branch the flows according to the split parameters $\langle\alpha_i\rangle$: $p_i^{out} = \alpha_i p^{in}$, where $(\sum_i \alpha_i) = 1$

Combine Rule : Whenever we see k (two or more) streams merging into one stream, we combine the flows of all the streams to compute the resultant combined flow: $\sum_i p_i^{in} = p_i^{out}$.

The analysis and the function of Split and Combine rules is very simple and straight-forward. In fact, the Clamp rule is the most intricate to analyze, and also most important to our model.

C Two-Parameter Models: A Village with Two Streams

C.1 Withdrawing Water is Return on Investment

Imagine a farm adjacent to a stream X. Let $\tilde{f}_X$ be the flow apparent to a farmer on the farm. If the farmer can withdraw a fraction γ of the flow apparent to it, then its return ($\tilde{R}_X$) is given by $\tilde{R}_X = \gamma\tilde{f}_X$, because a fraction γ is removed from the stream X with flow $\tilde{f}_X$ virtual to the farmer. The outgoing flow ($\tilde{f}_X^{out}$) will be $\tilde{f}_X - \tilde{R}_X = \tilde{f}_X^{out} = (1-\gamma)\tilde{f}_X$. We state these results as a lemma to facilitate references to the formulae.

Lemma C.1.1 *For $\tilde{f}_X, \tilde{f}_X^{out}, \tilde{R}_X$, and γ, defined above: $\tilde{R}_X = \gamma\tilde{f}_X$, and $\tilde{f}_X^{out} = (1-\gamma)\tilde{f}_X$.*

If f_X is a sum of several components, then the corollary below follows from the above lemma, considering that there is simple linear relationship between f_X and its components $g_{X,k}$.

Corollary C.1.1 *If $f_X = \sum_k g_{X,k}$, then for each k: $\tilde{g}_{X,k}^{out} = (1-\gamma)\tilde{g}_{X,k}$.*

When we incorporate dynamic adjustment, the return from stream X to a farmer on an adjacent farm will be given by the following lemma:

Lemma C.1.2 *The return from stream X is:* $\tilde{R}_X = (1 - e^{-\gamma})\tilde{f}_X$. *The outgoing flow will be* $\tilde{f}_X - \tilde{R}_X$: $\tilde{f}_X^{out} = e^{-\gamma}\tilde{f}_X$.

Proof: The rate of flow change is equal to the rate of water withdrawal, i.e. $df/d\tau = -\gamma f$ where $f(\tau)$ is the level of stream i, at instant τ, virtual to the farmer (with initial condition $f(0) = \tilde{f}_X$). This is an exact ordinary differential equation that can be written as $\frac{df}{f} = -\gamma d\tau$. Integrating both sides we get: $\ln(f) = -\gamma\tau + c$. When we exponentiate both sides and plug in the initial conditions, we get $f(\tau) = f(0)e^{-\gamma\tau}$. Consequently, for $\tau = 1$, we get $\tilde{f}_X^{out} = e^{-\gamma}\tilde{f}_X$, and $\tilde{R}_X = (1 - e^{-\gamma})\tilde{f}_X$ (since $\tilde{R}_X = \tilde{f}_X - \tilde{f}_X^{out}$). The lemma follows. □

If f_X consists of several components, then the corollary below follows from the above lemma, considering that there is simple linear relationship between f_X and its components $g_{X,k}$.

Corollary C.1.2 *If* $f_X = \sum_k g_{X,k}$, *then for each* k: $\tilde{g}_{X,k}^{out} = e^{-\gamma}\tilde{g}_{X,k}$.

Our model's *continuous* adjustment is slightly different from the *continual* adjustment that occurs in actual investment markets, but our *continuous* adjustment assumption facilitates the modeling, without substantially deviating from reality.

C.2 Our Two-Parameter Stream Model

We will first look at simplest case of two streams A and B, with cryptographic pseudo-random variables $\tilde{f}_A$ and $\tilde{f}_B$ denoting the respective flows. Each farmer has rights to withdraw a fraction α of water from the stream it chooses to tap, where α corresponds to the investment resources in capital markets. A farmer has to choose an apportioning strategy by selecting an x such that αx fraction will be withdrawn from Stream A, and $\alpha(1-x)$ fraction will be withdrawn from Stream B. Optimal strategy selects x to maximize the amount of water acquired. We can disallow short-selling by requiring $0 \le x \le 1$.

As described in § 3.1 and § 3.2, we develop two variations of the two-parameter multiple-player model:

Initial Level Model: This model computes the return for a farmer based on initial level of the stream for that farmer. Thus, if farmer taps a stream of level $\tilde{f}_X$ at the "rate" α, then the return would be $\alpha\tilde{f}_X$.

Dynamic Adjustment Model: This model computes the return for a farmer based on instantaneous the stream level for that farmer. Accordingly, the stream's level is dynamically adjusted to reflect the continuous change in level as it is tapped. As we prove in Lemma C.1.2, if farmer taps a stream of level $\tilde{f}_X$ at the "rate" α, then the return would be $(1 - e^{-\alpha})\tilde{f}_X$.

Suppose $\mathcal{C}(k)$ is the computational power of the player. Let each cryptographic random variable, $\tilde{f}_A$ and $\tilde{f}_B$, consist of two components, one $\mathcal{C}(k)$-predictable and one $\mathcal{C}(k)$-unpredictable. In other words, $\tilde{f}_A = p_A + \tilde{q}_A$, and $\tilde{f}_B = p_B + \tilde{q}_B$, where the player has computational power $\mathcal{C}(k)$, and p_A as well as p_B are $\mathcal{C}(k)$-predictable, while $\tilde{q}_A$ as well as $\tilde{q}_B$ are $\mathcal{C}(k)$-unpredictable random variables. We initially assume that there is no correlation between the $\mathcal{C}(k)$-predictable and $\mathcal{C}(k)$-unpredictable components.

We study four possible paradigms depending on the characteristics the unpredictable components ($\tilde{q}$). Recall that the optimal strategy of the farmer lies in maximizing $\alpha_i(x_i\tilde{f}_A^i + (1-x_i)\tilde{f}_B^i)$ (for the initial level model), or maximizing $(1 - e^{\alpha_i x_i})\tilde{f}_A^i + (1 - e^{\alpha_i(1-x_i)})\tilde{f}_B^i$ (for the dynamic adjustment model).

M_1 *The* $\mathcal{C}_i(k)$*-unpredictable elements are negligibly small relative to the* $\mathcal{C}_i(k)$*-predictable elements, i.e.* $\tilde{q}_A^i$ *and* $\tilde{q}_B^i$ *are negligibly small compared to* p_A^i *and* p_B^i*, respectively. Furthermore, both* $\tilde{q}_A^i$ *and* $\tilde{q}_B^i$ *are negligibly small compared to* $|p_A^i - p_B^i|$.

Since $\tilde{q}_A^i$ and $\tilde{q}_B^i$ are negligible, the farmers can ignore them in their computations of optimal strategy, and base their formulation of optimal strategy solely on the values of p_A^i and p_B^i.

Initial Level Model (ILM$_1$): To maximize its expected return, ith farmer (in the initial level model) needs to maximize $\alpha_i(x_i p_A^i + (1-x_i)p_B^i)$. Since α_i is inconsequential to maximization, the ith player must maximize the following expression with respect to x_i: $max_{x_i}[x_i p_A^i + (1-x_i)p_B^i]$.

Dynamic Adjustment Model (DAM$_1$): To maximize its expected return, ith farmer must maximize the following expression with respect to x_i: $max_{x_i}[(1 - e^{-\alpha_i x_i})p_A^i + (1 - e^{-\alpha_i(1-x_i)})p_B^i]$.

M_2 *The* $\mathcal{C}_i(k)$*-unpredictable elements are known constants, i.e.* $\tilde{q}_A^i$ *and* $\tilde{q}_B^i$ *are known constants* Q_A^i *and* Q_B^i*, respectively.*

Since $\tilde{q}_A^i = Q_A^i$ and $\tilde{q}_B^i = Q_B^i$, the farmer can substitute these values in its expected return that it wants to maximize. The farmers can base their formulation of optimal strategy solely on the values of $p_A^i + Q_A^i$ and $p_B^i + Q_B^i$.

Initial Level Model (ILM$_2$): Optimal strategy for ith player must maximize the following expression with respect to x_i: $max_{x_i}[x_i(p_A^i + Q_A^i) + (1-x_i)(p_B^i + Q_B^i)]$.

Dynamic Adjustment Model (DAM$_2$): Optimal strategy for ith player must maximize the following expression with respect to x_i: $max_{x_i}[(1 - e^{-\alpha_i x_i})(p_A^i + Q_A^i) + (1 - e^{-\alpha_i(1-x_i)})(p_B^i + Q_B^i)]$.

M$_3$ *The expected values of random variables $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are known constants, i.e. $E[\tilde{q}^i_A] = Q^i_A$ and $E[\tilde{q}^i_B] = Q^i_B$.*

We take advantage of the fact that expectation of expectation of random variable is just its expectation (i.e. $E[E[\tilde{X}]] = E[\tilde{X}]$). Hence, we can substitute $\tilde{q}^i_A = Q^i_A$ and $\tilde{q}^i_B = Q^i_B$, and apply the analysis of M$_2$ above.

Initial Level Model (ILM$_3$): Optimal strategy for ith player must maximize the following expression with respect to x_i: $max_{x_i}[x_i(p^i_A + Q^i_A) + (1 - x_i)(p^i_B + Q^i_B)]$.

Dynamic Adjustment Model (DAM$_3$): Optimal strategy for ith player must maximize the following expression with respect to x_i: $max_{x_i}[(1 - e^{-\alpha_i x_i})(p^i_A + Q^i_A) + (1 - e^{-\alpha_i(1-x_i)})(p^i_B + Q^i_B)]$.

M$_4$ *There are no restrictions on the $C_i(k)$-unpredictable elements, except that their upper and lower bounds are known, i.e. $\tilde{q}^i_A$ is bounded above and below by L^i_A and U^i_A, whereas $\tilde{q}^i_B$ is bounded above and below by L^i_B and U^i_B. $\tilde{q}^i_A$ and $\tilde{q}^i_B$ are not necessarily constants, and they are not necessarily negligible compared to p^i_A and p^i_B, respectively.*

In this case the ith farmer's optimal strategy is to compute x_i so as to maximize the worst case return in view of the lower and upper bounds.

Initial Level Model (ILM$_4$): To maximize its expected return, ith farmer (in the initial level model) needs to maximize the following expression with respect to x_i within the constraints that $L^i_A \le \tilde{q}^i_A \le U^i_A$ and $L^i_B \le \tilde{q}^i_B \le U^i_B$: $max_{x_i}[x_i(p^i_A + \tilde{q}^i_A) + (1 - x_i)(p^i_B + \tilde{q}^i_B)]$.

Dynamic Adjustment Model (DAM$_4$): To maximize its expected return, ith farmer (in the initial level model) needs to maximize the following expression with respect to x_i within the constraints that $L^i_A \le \tilde{q}^i_A \le U^i_A$ and $L^i_B \le \tilde{q}^i_B \le U^i_B$: $max_{x_i}[(1 - e^{-\alpha_i x_i})(p^i_A + \tilde{q}^i_A) + (1 - e^{-\alpha_i(1-x_i)})(p^i_B + \tilde{q}^i_B)]$.

Since M$_1$, M$_2$, and M$_3$ can be analyzed using simpler techniques, we say that they satisfy condition C*. On the other hand, M$_4$ requires different analysis, and does not satisfy condition C*.

C.3 Diminishing Disparity between $\mathcal{C}(k)$-predictable Components (Initial Level Model)

We shall first analyze the ILM$_1$, and observe that the player exploits any disparity between $\mathcal{C}(k)$-predictable components of two streams. Furthermore, the disparity between $\mathcal{C}(k)$-predictable components of the two streams decreases when a player exploits it to maximize its return (unless the two components are already very close together).

Consider the two-parameter stream model with $\tilde{f}_A$ and $\tilde{f}_B$ being the flow of the streams A and B. Let $\tilde{f}_A = p_A + \tilde{q}_A$ and $\tilde{f}_B = p_B + \tilde{q}_B$ being the flow of the two streams, and let $\tilde{f}'_A = p'_A + \tilde{q}'_A$ and $\tilde{f}'_B = p'_B + \tilde{q}'_B$ be the respective flow of the two streams after one player has employed optimal strategy to withdraw water. The following two lemmas and one theorem are important for our initial level model, with $\tilde{f}_A = p_A + \tilde{q}_A$, $\tilde{f}_B = p_B + \tilde{q}_B$, $\tilde{f}'_A = p'_A + \tilde{q}'_A$, $\tilde{f}'_B = p'_B + \tilde{q}'_B$, α, and x defined above.

Lemma C.3.1 *$\tilde{f}'_A = \alpha x \tilde{f}_A$, $p'_A = \alpha x p_A$, and $\tilde{q}'_A = \alpha x \tilde{q}_A$.Similarly, $\tilde{f}'_B = \alpha(1 - x)\tilde{f}_B$, $p'_B = \alpha(1 - x)p_B$, and $\tilde{q}'_B = \alpha(1 - x)\tilde{q}_B$.*

Proof: The lemma follows trivially from Lemma C.1.1 and Corollary C.1.1. □

Lemma C.3.2 *If short-selling is disallowed by requiring $0 \le x \le 1$, then the optimal strategy is to select:* $x = 1$ if $p_A > p_B$; $x = \frac{1}{2}$ if $p_A = p_B$; $x = 0$ if $p_A < p_B$

Proof: The player aims to maximize $\alpha x p_A + \alpha(1 - x)p_B$ with respect to parameter x (with $0 \le x \le 1$). After removing irrelevant constants, this amounts to maximizing $x(p_A - p_B)$. If $p_A > p_B$ or $p_A < p_B$, then obviously $x = 1$ and $x = 0$, respectively. When $p_A = p_B$, any x would do, but $x = \frac{1}{2}$ is selected since it is the most symmetric choice. □

Theorem C.3.1 *If the player pursues optimal strategy (without short-selling, i.e. $0 \le x \le 1$):*

If $|p_A - p_B| = 0$, then $|p'_A - p'_B| = 0 = |p_A - p_B|$.

If $|p_A - p_B| > 0$ then $|p'_A - p'_B| < |p_A - p_B|$ provided if $p_A \ne p_B$ then $(1 - \alpha)larger(p_A, p_B) \ge smaller(p_A, p_B)$.

Proof: If $|p_A - p_B| = 0$, then $p_A = p_B$. Therefore, $x = \frac{1}{2}$ is chosen. Hence, $p'_A = \frac{\alpha p_A}{2} = \frac{\alpha p_B}{2} = p'_B$. Consequently, $|p'_A - p'_B| = 0 = |p_A - p_B|$.

If $|p_A - p_B| > 0$ then $p_A \ne p_B$, and without loss of generality we can assume $p_A > p_B$. Consequently, the player can maximize $\alpha x p_A + \alpha(1 - x)p_B$ (with respect to parameter x) by choosing $x = 1$. Thus, $p'_A = (1 - \alpha)p_A$ and $p'_B = p_B$. $(1 - \alpha)larger(p_A, p_B) \ge smaller(p_A, p_B)$ implies that $(1 - \alpha)p_A - p_B \ge 0$. Furthermore we know that $(1 - \alpha)p_A < p_A$, and $p_A - p_B \ge 0$. Using these three inequalities, we can derive the following: $|p'_A - p'_B| = |(1 - \alpha)p_A - p_B| = (1 - \alpha)p_A - p_B < p_A - p_B = |p_A - p_B|$. Therefore, $|p'_A - p'_B| < |p_A - p_B|$. The theorem follows. □

The above analysis can be extended to paradigm ILM$_2$ and ILM$_3$ of § C.3 by substituting $p^i_A + Q^i_A$ for p^i_A, and substituting $p^i_B + Q^i_B$ for p^i_B.

C.4 Diminishing Disparity between $\mathcal{C}(k)$-predictable Components (Dynamic Model)

In our analysis of dynamic paradigm in DAM$_1$, we shall see that we do not need the condition "if $p_A \neq p_B$ then $(1-\alpha)larger(p_A, p_B) \geq smaller(p_A, p_B)$" of Theorem C.3.1, which restricts our analysis when there is little virtual disparity between Streams A and B. We observe that the disparity between $\mathcal{C}(k)$-predictable components of the two streams decreases.

The following two lemmas and one theorem are important for our model incorporating dynamic adjustment, with $\tilde{f}_A = p_A + \tilde{q}_A$, $\tilde{f}_B = p_B + \tilde{q}_B$, $\tilde{f}'_A = p'_A + \tilde{q}'_A$, $\tilde{f}'_B = p'_B + \tilde{q}'_B$, α, and x defined earlier.

Lemma C.4.1 *$\tilde{f}'_A = e^{-\alpha x}\tilde{f}_A$, $p'_A = e^{-\alpha x}p_A$, and $\tilde{q}'_A = e^{-\alpha x}\tilde{q}_A$. Similarly, $\tilde{f}'_B = e^{-\alpha x}\tilde{f}_B$, $p'_B = e^{-\alpha x}p_B$, and $\tilde{q}'_B = e^{-\alpha x}\tilde{q}_B$.*

Proof: The lemma follows trivially from Lemma C.1.2 and Corollary C.1.2. □

Lemma C.4.2 *The optimal strategy is to select $x = \frac{\alpha+\log(p_A/p_B)}{2\alpha}$. However, if short-selling is disallowed (by requiring $0 \leq x \leq 1$), then the player chooses $x = 0$ if $\frac{\alpha+\log(p_A/p_B)}{2\alpha} \leq 0$, and $x = 1$ if $\frac{\alpha+\log(p_A/p_B)}{2\alpha} \geq 1$.*

Proof: Suppose, we were to apportion between two flows p_A and p_B, such that x is the fraction of time used to extract from Stream A, and $1-x$ is the fraction of time used to extract from Stream B. The optimal strategy for the player amounts maximizing its expected return, which means choosing x to maximize $R^i_P = (1-e^{-\alpha x})p_A + (1-e^{-\alpha(1-x)})p_B$. We do this setting the derivative of R^i_P with respect to x equal to zero, i.e. $\frac{dR^i_P}{dx} = \alpha e^{-\alpha x}p_A - \alpha e^{-\alpha(1-x)}p_B = 0$. Simplifying, we get $e^{2\alpha x - \alpha} = \frac{p_A}{p_B}$. Taking logarithm of both sides, we get $2\alpha x - \alpha = \log(\frac{p_A}{p_B})$. Solving for x, we get $x = \frac{\alpha+\log(p_A/p_B)}{2\alpha}$.

If we disallow short-selling, we check the boundary conditions ($x = 0$ and $x = 1$), and limit our result to the maximum in that region: If $0 \leq \frac{\alpha+\log(p_A/p_B)}{2\alpha} \leq 1$ then $x = \frac{\alpha+\log(p_A/p_B)}{2\alpha}$, otherwise $x = 0$ if $p_A < p_B$, and $x = 1$ if $p_B < p_A$. If short-selling is allowed $x = \frac{\alpha+\log(p_A/p_B)}{2\alpha}$ is the optimal strategy. The lemma follows. □

Theorem C.4.1 *If the player pursues optimal strategy then $|p'_A - p'_B| < |p_A - p_B|$ if $|p_A - p_B| > 0$ and $|p'_A - p'_B| = |p_A - p_B|$ if $|p_A - p_B| = 0$*

Proof: If $|p_A - p_B| = 0$ then $p_A = p_B$, and hence by Lemma C.4.2 $x = \frac{1}{2}$. Consequently, $p'_A = p'_B$, and that means $|p'_A - p'_B| = 0 = |p_A - p_B|$.

If $|p_A - p_B| \neq 0$ then, without loss of generality, we can assume $p_A > p_B$. Consequently, the player can maximize $(1-e^{-\alpha x})p_A + (1-e^{-\alpha(1-x)})p_B$ (with respect to parameter x) by choosing x according to Lemma C.4.2. We can formally split the proof into two mutually exclusive and collectively exhaustive cases:

Case 1 : $e^{-\alpha}p_A \geq p_B \implies$ Optimal strategy (according to Lemma C.4.2) will select $x = 1$, and thus $R_P = (1-e^{-\alpha})(p_A + \tilde{q}_A)$. Furthermore, $\tilde{f}'_A = e^{-\alpha}\tilde{f}_A$, $\tilde{f}'_B = \tilde{f}_B$, and $p'_A = e^{-\alpha}p_A$, $p'_B = p_B$. Consequently, since $e^{-\alpha}(p_A) - p_B \geq 0$, $e^{-\alpha}(p_A) < p_A$, and $p_A - p_B \geq 0$, we can derive the following: $|p'_A - p'_B| = |e^{-\alpha}(p_A) - p_B| = e^{-\alpha}(p_A) - p_B < p_A - p_B = |p_A - p_B|$. Thus, $|p'_A - p'_B| < |p_A - p_B|$. In fact, $\frac{|p'_A - p'_B|}{|p_A - p_B|} = \frac{e^{-\alpha}(p_A) - p_B}{p_A - p_B} < 1$.

Case 2: $e^{-\alpha}(p_A) < p_B \implies$ The optimal strategy will to set $x = \frac{\alpha+\ln(p_A/p_B)}{2\alpha}$ according to Lemma C.4.2. This value is between 0 and 1, inclusive. When we plug it into $p'_A = e^{-\alpha x}p_A$ and $p'_B = e^{-\alpha(1-x)}p_B$, we get $p'_A = p'_B$ after a few lines of algebra. Consequently, $|p'_A - p'_B| = 0 < |p_A - p_B|$.

The theorem follows since the aforementioned cases are collectively exhaustive. □

A stronger result holds if short-selling is allowed.

Theorem C.4.2 *For p'_A and p'_B defined above, if the player pursues optimal strategy and short-selling is allowed then $|p'_A - p'_B| = 0$.*

Proof: According to Lemma C.4.2, the first player to move will choose $x = \frac{\alpha+\log(p_A/p_B)}{2\alpha}$. When this value is substituted for x in formulae for p'_A and p'_B, we get $p'_A = p'_B$. The theorem follows. □.

The above analysis can be extended to paradigms DAM$_2$ and DAM$_3$ by substituting $p^i_A + Q^i_A$ for p^i_A, and substituting $p^i_B + Q^i_B$ for p^i_B.

D Our Two Parameter-Multiple Player Model

In this section, we extend our Two Parameter Stream Model to accommodate several players who take turns to move, in some arbitrary order. The basic Two parameter-Multiple Player model is illustrated in Figure 2 earlier. It consists of a row of several farmers stationed on a valley banking on two streams. Farmer i has rights to withdraw a fraction α_i of water by splitting α_i into $\alpha_i x_i$ and $\alpha_i(1-x_i)$, where $\alpha_i x_i$ is the fraction withdrawn from Stream A, and $\alpha_i(1-x_i)$ is the fraction withdrawn from Stream B. Each farmer distributes its rights over two streams so as to maximize the amount of water it acquires.

Suppose the initial levels of two streams, A and B, are $\tilde{f}_A$ and $\tilde{f}_B$, respectively. We let $\tilde{f}_A^i$ and $\tilde{f}_B^i$ denote the levels of two streams (A and B, respectively) just before the water is accessible to the ith farmer. Consequently, $\tilde{f}_A^1 = \tilde{f}_A$ and $\tilde{f}_B^1 = \tilde{f}_B$.

D.1 Two Parameter-Multiple Players (Initial Level) Model

The return for the ith farmer is the amount of water it withdraws, which is $\alpha_i(x_i\tilde{f}_A^i + (1-x_i)\tilde{f}_B^i)$ units. This leaves Stream A's level for the next farmer down the stream at $\tilde{f}_A^{i+1} = (1-\alpha_i x_i)\tilde{f}_A^i$, and Stream B's level at $\tilde{f}_B^{i+1} = (1-\alpha_i(1-x_i))\tilde{f}_B^i$ units (by simple application and extension of Lemma C.1.1). For $\tilde{f}_A^i = p_A^i + \tilde{q}_A^i$, $\tilde{f}_B^i = p_B^i + \tilde{q}_B^i$, α_i, and x_i defined above:

Theorem D.1.1 *Assuming that all players pursue optimal strategy:*

If $|p_A^i - p_B^i| > 0$ *then* $|p_A^n - p_B^n|$ *decreases monotonically as* n *increases, until* $(1-\alpha_n)$ larger$(p_A^n, p_B^n) <$ smaller(p_A^n, p_B^n). *If* $|p_A^i - p_B^i| = 0$ *then it remains* 0.

Proof: If $|p_A^i - p_B^i| = 0$, then it remains 0, by application of Theorem C.3.1.

If $|p_A^i - p_B^i| > 0$, then, according to Theorem C.3.1, $|p_A^{i+1} - p_B^{i+1}| < |p_A^i - p_B^i|$ if $(1-\alpha_i)$ larger$(p_A^i, p_B^i) \geq$ smaller(p_A^i, p_B^i). By induction, $|p_A^n - p_B^n|$ decreases monotonically as n increases, until $(1-\alpha_n)$ larger$(p_A^n, p_B^n) <$ smaller(p_A^n, p_B^n). The theorem follows. □

If $|p_A^i - p_B^i| > 0$, then we have a potentially non-converging situation, because $|p_A^n - p_B^n|$ can easily get into a non-converging case. In particular, we examine the case where all players have the same access rights ($\alpha = \alpha_i$ for all i).

Corollary D.1.1 *Consider the case where* $|p_A^i - p_B^i| > 0$, *and the withdrawal rights for all players are the same* ($\alpha = \alpha_i$). *If there exists* n *such that* $(1-\alpha_n)$ larger$(p_A^n, p_B^n) <$ smaller(p_A^n, p_B^n), *then* $|p_A^m - p_B^m| > 0$ *for all finite* m.

Proof: If $(1-\alpha)$ larger$(p_A^i, p_B^i) <$ smaller(p_A^i, p_B^i)] then $n \geq 0$, we have larger$(p_A^{i+2n}, p_B^{i+2n}) = (1-\alpha)^n$larger$(p_A^i, p_B^i)$, and larger$(p_A^{i+1+2n}, p_B^{i+1+2n}) = (1-\alpha)^n$smaller$(p_A^i, p_B^i)$. Since $0 < (1-\alpha) < 1$, we can observe the repeated oscillations, and the corollary follows. □.

Paradigms ILM_2 and ILM_3 satisfy condition C*, and the above analysis can be extended to them, and they can be analyzed by substituting $p_A^i + Q_A^i$ for p_A^i, and substituting $p_B^i + Q_B^i$ for p_B^i. ILM_4 does not satisfy condition C*, and we cannot readily adapt the above analysis for it.

D.2 Shortcomings of Two Parameter-Multiple Players (Initial Level) Model

In this subsection, we scrutinize some shortcomings of our Two Parameter-Multiple Players (Initial Level) Model. Subsequently, we enhance our the model to incorporate more realistic features. There are two major problems with the treatment in previous subsection.

- It is unrealistic to assume that a security's price (or a stream's level) will not change during the processing of major transactions.
- We are caught in repeated oscillations instead of complete convergence to 0, when the virtual flows of the two streams get very close to each other.

We propose a model in which the amount of water withdrawn from a stream by any farmer depends on the stream's *instantaneous* level, rather than the stream's level at the time it becomes accessible to the farmer (as assumed in the formulation in § D.1). This is analogous to the stock market situation in which if an investor decides to sell 100,000 shares of IBM on a Monday morning, then the price of the stock is going to vary continually while this big volume is being sold. We introduce, formalize, and analyze a dynamic case in which the flow of the stream is continuously adjusted to successive water withdrawals, just like stock prices adjust to a sizable transaction.

There are two main causes for variation in price during any major transaction:

1. Any major buy or sell order is going to affect the supply-demand equilibrium, and consequently influence the price of a security.
2. A large buy (or sell) order cannot be processed until they are matching sell (or buy, respectively) orders. Since this may take time, the price is likely to fluctuate during the process of any large order.

The cause of repeated oscillations (as noted in third row in Table 3) is the fact that where there is only a slight difference between two stream's predictable components, the player would choose to invest all its resources (i.e. allocate all its withdrawal rights) to the one with slightly higher level. In turn, when it draws a return from the slightly higher level, it can increase magnitude of the disparity between the two streams, as exhibited by the change from item 4 to item 5 in Table 3. It is obvious that the difference between two streams' $\mathcal{C}(k)$-predictable components increases from item 4 to item 5. Subsequently, we observe oscillations which mitigate convergence.

D.3 Two Parameter-Multiple Players (with Dynamic Adjustment)

In the dynamic case, the ith player can withdraw α_i from *instantaneous* flow of the streams (which is dynamically adjusting to reflect any water that has been taken out). The rapid adjustment of flow to withdrawals is more realistic, and is analogous to adjustment of stock prices to a sizable transaction order.

The return for the ith farmer is the amount of water it withdraws, which is $e^{-\alpha_i x_i}\tilde{f}^i_A + e^{-\alpha_i(1-x_i)}\tilde{f}^i_B$ units. This leaves Stream A's level for the next farmer down the stream at $\tilde{f}^{i+1}_A = (1-e^{-\alpha_i x_i})\tilde{f}^i_A$, and Stream B's level at $\tilde{f}^{i+1}_B = (1-e^{-\alpha_i(1-x_i)})\tilde{f}^i_B$ units (by simple application and extension of Lemma C.1.2). For $\tilde{f}^i_A = p^i_A + \tilde{q}^i_A$, $\tilde{f}^i_B = p^i_B + \tilde{q}^i_B$, α_i, and x_i defined above:

Theorem D.3.1 *If all the players pursue optimal strategy then: $|p^i_A - p^i_B|$ decreases monotonically until it reaches zero.*

Proof: If $|p^1_A - p^1_B|$ is zero, then by Theorem C.3.1 it stays at zero, and this theorem follows trivially.

If $|p^1_A - p^1_B| > 0$ then by applying Theorem C.3.1, $|p^{i+1}_B - p^{i+1}_B| < |p^i_A - p^i_B|$. Consequently, $\{|p^i_A - p^i_B|\}$ forms a monotonically decreasing sequence while it is positive, and is bounded below by 0. Therefore, for some $i > n$, $|p^i_A - p^i_B| = 0$. The theorem follows. □

We have already observed that $|p^i_A - p^i_B|$ diminishes rapidly until it reaches *zero*. However, it remains to be gauged how fast it approaches zero, i.e. find $N(\epsilon)$ such that $|p^i_A - p^i_B| < \epsilon$ for $i > N(\epsilon)$. Without loss of generality, assume $p^i_A \geq p^i_B$. If we further assume that all players' forecasting power and withdrawal rights are the same then: There is an n such that $e^{-\alpha(n+1)}p^1_A < p^1_B < e^{-\alpha n}p^1_A$, and that implies that $p^i_A = p^i_B$ for $i > n$. Therefore, we have the following Corollary:

Corollary D.3.1 *For notation defined above, p^{n+1}_B becomes equal to p^{n+1}_A for the first time for $n = \lceil \frac{\log(\frac{larger(p^1_A, p^1_B)}{smaller(p^1_A, p^1_B)})}{\alpha} \rceil$. Subsequently, $p^i_A = p^i_B$ for $i > n$.*

Without loss of generality, assume $p^i_A \geq p^i_B$. If all α_i's are not necessarily identical then: There is an n such that $e^{(\sum_i^{n+1}) - \alpha_i} p^1_A < p^1_B < e^{(\sum_i^{n}) - \alpha_i} p^1_A$, and that implies that $p^i_A = p^i_B$ for $i > n$. Therefore, we have the following Corollary:

Corollary D.3.2 *For notation defined above, p^{n+1}_B becomes equal to p^{n+1}_A for the first time for the smallest n such that $\sum_i^{n+1} \alpha_i \geq \lceil \log(\frac{larger(p^1_A, p^1_B)}{smaller(p^1_A, p^1_B)}) \rceil$. Subsequently, $p^i_A = p^i_B$ for $i > n$.*

The consequences of Theorem D.3.1 are summarized in the following Corollary:

Corollary D.3.3 *The advantage derived from pursuing optimal strategy diminishes as the disparity between the predictable components approaches zero. Consequently, as the is disparity between $C(k)$-predictable components diminishes, the advantage derived from predicting decreases, until the optimal strategy has no advantage over an indifferent strategy.*

The following points are worth observing:

1. There is a near linear decay of the larger $C(k)$-predictable until it equals the smaller $C(k)$-predictable component.
2. Decay of the difference $|p^i_A - p^i_B|$ is a bounded above by a near linear monotonic function.
3. $C(k)$-unpredictable random variables retain their original distribution with a scaled mean and variance.

Aforementioned cases DAM₂ and DAM₃ satisfy condition C*, and the above analysis can be extended to them by substituting $p^i_A + Q^i_A$ for p^i_A, and substituting $p^i_B + Q^i_B$ for p^i_B. DAM₄ does not satisfy condition C*, and we cannot readily adapt the above analysis for it.

D.4 Selling Short and Going Long

In most security markets, investors can issue securities as well. This is known as short-selling, and is accomplished by selling a promise to pay at a later time, whatever the market value of the security is at that time, plus any dividends paid by the firm during that time. If the market is assured that the investor can deliver on its promise, then it will pay the investor the current price of share for every share he issues. The investors can then use the proceeds from short-selling to invest in other securities.

Short-selling on a Security A involves virtually borrowing shares from some owner at time t, and agreeing to return them at time $t + t_0$. The investor then immediately sells these borrowed shares in the market. At time $t + t_0$, the investor repurchases the shares of A from the market and returns them to the lender at that time. When an investor borrows some securities from another investor and sells them in the market, he is said to have a "short" position in those shares. On the other hand, owning a security is termed as "long" position. In terms of our stream-parameter model, a player may go "short" by selling its rights to withdraw water from Stream A for rights to withdraw from Stream B.

Short-selling is a tool employed by investors to increase their leverage by allowing one investor with limited funds to be more influential in the market than it could have been otherwise. Using short-selling, one player can have an impact equivalent to several investors in the market.

In the preceding subsections, we have assumed that x, the fraction invested in A, was always between 0 and 1 inclusive. Consequently, $(1-x)$, the fraction invested in B was also always between 0 and 1 inclusive. The quantitative effect of short-selling, is that the value of x is allowed to be more than 1 or less than 0. Thus, if $x < 0$, then Stream A is sold short, and if $x > 1$, then Stream B is sold short. Short-selling does not integrate with ILMs (Initial Level Models) since a player can extract unlimited return because the levels (prices) are insensitive to large transactions.

If short-selling is allowed, it takes only one farmer's optimal action before we reach the steady state of $|p^i_A - p^i_B| = 0$. In the running example, for $\alpha_i = 0.03$, we get $x = 1.8896935$, $p^2_A = p^2_B = 94.488618$.

Theorem D.4.1 *If the player pursues optimal strategy and short-selling is allowed then* $|p'_A - p'_B| = 0$.

Corollary D.4.1 *Regardless of the initial conditions, short-selling allows one farmer to achieve what takes several farmers to achieve:*

1. *Reduce the difference between the* $\mathcal{C}(k)$*-predictable elements.*
2. *There is no disparity between the* $\mathcal{C}(k)$*-predictable return of all farmers except the first one.*

Cases DAM$_2$ and DAM$_3$ of § C.2 can be analyzed by substituting $p^i_A + Q^i_A$ for p^i_A, and substituting $p^i_B + Q^i_B$ for p^i_B.

D.5 Analyzing Paradigms which Do Not Satisfy Condition C*

We have seen that ILM$_4$ and DAM$_4$ do not satisfy condition C*, so we can't analyze them using the methods employed for cases which satisfy condition C* (i.e. M$_1$, M$_2$, and M$_3$). In this subsection, we develop methods for analyzing paradigms ILM$_4$. Our analysis assumes that short-selling is disallowed, as it is for all ILMs. We conclude by removing that restriction, and generalizing our analysis to cases where short-selling is restricted.

Proposition D.5.1 *We analyze the feasibility of any strategy by its worst case performance with respect to the "indifferent strategy" in which an "indifferent farmer" splits its rights* equally *among all accessible streams. Quantitative consequence of the indifferent strategy is that we always chose* $x = \frac{1}{2}$ *in the two-parameter stream model.*[9]

Recall that L_A and U_A are lower and upper bounds of $\tilde{q}_A$, respectively, and L_B and U_B are lower and upper bounds of $\tilde{q}_B$, respectively.

The feasibility of a strategy measured is return for indifferent strategy is measured by its worst case performance with respect to the "indifferent strategy" as stated in Proposition D.5.1.

Theorem D.5.1 *The optimal strategy (for the worst case analysis) selects:*
$\mathbf{x} = 1$ *if* $L_A \geq p_B + U_B - p_A$,
$\mathbf{x} = 0$ *if* $L_B \geq p_A + U_A - p_B$,
$\mathbf{x} = \frac{1}{2}$ *otherwise.*

Proof: The feasibility equation can be obtained by computing the difference between a strategy and the "indifferent" strategy. Hence the most feasible value of x can be found by minimizing the $min_x[(x-\frac{1}{2})((p_A + \tilde{q}_A) - (p_B + \tilde{q}_B))]$ with respect to x, subject to the constraints $L_A \leq \tilde{q}_A \leq U_A$, and $L_B \leq \tilde{q}_B \leq U_B$.

If the player selects $x > \frac{1}{2}$, the worst case oracle will select $\tilde{q}_A = L_A$ and $\tilde{q}_B = U_B$, so as to minimize the player's return. Hence, selecting $x > \frac{1}{2}$ would be profitable only if $L_A \geq p_B + U_B - p_A$ (since this condition implies $(p_A + \tilde{q}_A) - (p_B + \tilde{q}_B) > 0$. Furthermore, since $L_A \geq p_B + U_B - p_A$ guarantees that $(p_A + \tilde{q}_A) - (p_B + \tilde{q}_B)$ is positive, the player will do best to select $x = 1$, the maximum allowed value of x.

Analogously, if the player selects $x < \frac{1}{2}$, the worst case oracle will select $\tilde{q}_A = U_A$ and $\tilde{q}_B = L_B$, so as to minimize the player's return. Hence, selecting $x < \frac{1}{2}$ would be profitable only if $L_B \geq p_A + U_A - p_B$ (since this condition implies $(p_A + \tilde{q}_A) - (p_B + \tilde{q}_B) < 0$. Furthermore, since $L_B \geq p_A + U_A - p_B$ guarantees that $(p_A + \tilde{q}_A) - (p_B + \tilde{q}_B)$ is negative, the player will do best to select $x = 0$, the minimum allowed value of x.

If neither of the two conditions are satisfied then to guard against a malicious oracle (which arranges $\tilde{q}_A$ and $\tilde{q}_B$ to produce the worst case), the player should choose $x = \frac{1}{2}$ to perform no worse than indifferent strategy. The theorem follows. □

E Multiple Parameter Portfolios

E.1 Our Extension of the Classical Two-parameter Model

Now we enhance the classical one-period, two-parameter model is due to Markowitz [Markowitz52, Markowitz59] by modeling multiple parameter portfolios with water acquisition from streams. At time $t = 1$, and Investor j

[9] When we have k streams, each stream is assigned $x_i = \frac{1}{k}$.

Farmer	Stream 1 q_1	p_1	Stream 2 q_2	p_2	Stream 3 q_3	p_3	$max\|p^j_h - p^j_i\|$	$\frac{max\|p^j_h - p^j_i\|}{max\, p^j_g}$	x^j_1	x^j_2	x^j_3
1	$\tilde{Q}+$	100	$\tilde{Q}+$	96	$\tilde{Q}+$	92	8.0000	0.08000	1	0	0
2	$0.9700\tilde{Q}+$	97	$\tilde{Q}+$	96	$\tilde{Q}+$	92	5.0000	0.05155	1	0	0
3	$0.9409\tilde{Q}+$	94.09	$\tilde{Q}+$	96	$\tilde{Q}+$	92	4.0000	0.04167	0	1	0
4	$0.9409\tilde{Q}+$	94.09	$0.9700\tilde{Q}+$	93.12	$\tilde{Q}+$	92	2.0900	0.02221	1	0	0
5	$0.9127\tilde{Q}+$	91.27	$0.9700\tilde{Q}+$	93.12	$\tilde{Q}+$	92	1.8527	0.01990	0	1	0
6	$0.9127\tilde{Q}+$	91.27	$0.9409\tilde{Q}+$	90.32	$\tilde{Q}+$	92	1.6800	0.01826	0	0	1
7	$0.9127\tilde{Q}+$	91.27	$0.9409\tilde{Q}+$	90.32	$0.97\tilde{Q}+$	89.24	2.0273	0.02221	1	0	0
8	$0.8853\tilde{Q}+$	88.53	$0.9409\tilde{Q}+$	90.32	$0.97\tilde{Q}+$	89.24	1.7907	0.01990	...	...	...
9	...	...	...	...	...	...	...	...	...	...	...
10	...	...	...	...	...	...	...	...	...	...	...

Table 5: Changes in Levels in Three-Parameter Stream Model

has wealth w^j_1 which he must allocate to consumption c^j_1 and investment $(w^j_1 - c^j_1)$ in some portfolio of securities. The consumption at time $t = 2$ is $\tilde{c}_2 = (w^j_1 - c^j_1)\tilde{R}^j_P$, such that $\tilde{R}^j_P$ is the return to investor j on portfolio P given by $\tilde{R}^j_P = \sum_i x^j_i \tilde{R}^j_i$, such that x^j_i is the fractional allocation to Stream i by Player j, and $\sum_i x^j_i = 1$ for every player j. Here, $\tilde{R}^j_i$ denotes returns on investment in Stream i, and x^j_i determines the distribution of investments across different streams. We initiate by analyzing development of optimal expected return strategy, and its effect on the efficiency and randomness of the market. We shall later discuss other classical issues like risk potential, market equilibrium, and correlation between returns on different resource allocation alternatives faced by the players. We refer the reader to Chapter 7 and 8 of Fama's book [Fama76] for background reading on these issues.

The basic three-parameter stream model with 8 farmers is shown in Figure 8. The figure readily generalizes to depict any number of streams. Initial level of stream i is denoted by $\tilde{f}_i$, and the level for player j is denoted by f^j_i respectively. Consequently, $f^1_i = \tilde{f}_i$.

Each player j has withdrawal rights expressed as a fraction α_j, which is analogous to financial resources in the investment market. α_j must be apportioned by a vector $\langle \alpha_j x^j_i \rangle$, where $\alpha_j x^j_i$ is the fraction allocated to to be withdrawn from Stream i. If we disallow short-selling then we require $0 \le x^j_i \le 1$ for all x^j_i, otherwise $x^j_i \in \Re$. The return of the jth farmer is given by $R^j_P = \alpha_j \sum_i (x^j_i \tilde{f}^j_i)$. The levels of stream for player $j+1$ (i.e. $\tilde{f}^{j+1}_i$) can be calculated from the levels of player j (i.e. $\tilde{f}^j_i$) by the formulae $\tilde{f}^{j+1}_i = (1 - \alpha_j x^j_i)\tilde{f}^j_i$.

Naturally, every farmer will try to optimize the expected amount of water it taps. Thus, the farmer j chooses $\langle x^j_i \rangle$ to maximize R^i_P, i.e. $max_{\langle x^j_i \rangle}[R^i_P] = max_{\langle x^j_i \rangle}[\alpha_j \sum_i (x^j_i \tilde{f}^j_i)]$

Suppose that each $\tilde{f}_i$ consists of components that can be predicted by the players with some given computational power, and not by any player with less than that computational power. We will first look at the simplest of these cases, where Player j has computational power $C_j(k)$, and $\tilde{f}^j_i = p^j_i + \tilde{q}^j_i$ such that p^j_i is $C_j(k)$-predictable, and $\tilde{q}^j_i$ is $C_j(k)$-unpredictable. Let us suppose that all players have exactly the same computational power, i.e. $C_j(k) = C(k)$ for all j.

Now, we proceed with some basic analysis of our model. Player j's return on some portfolio P (i.e. allocation P) is given by the following function: $R^j_P = \alpha_j(\sum_i x^j_i \tilde{f}^j_i) = \alpha_j(\sum_i x^j_i p^j_i + \sum_i x^j_i \tilde{q}^j_i)$ Consequently, $E[R^j_P] = \alpha_j(\sum_i x^j_i E[\tilde{f}^j_i]) = \alpha_j(\sum_i x^j_i p^j_i + \sum_i x^j_i E[\tilde{q}^j_i])$.

We shall analyze ILM₁ and DAM₁ to build our intuition before we delve deeper. ILM₁ assumes $\tilde{q}^j_i << p^j_i$ for all streams i, and $\tilde{q}^j_i << |p^j_k - p^j_l|$ for any three streams i, k, and l. Hence, jth farmer's optimal strategy to maximize R^i_P effectively means choosing $\langle x^j_i \rangle$ so as to maximize $max_{\langle x^j_i \rangle}[\sum_i (x^j_i p^j_i)]$.

Suppose that initially $\tilde{f}_1 = \tilde{f}^1_1 = 100 + \tilde{q}^1_1$, $\tilde{f}_2 = \tilde{f}^1_2 = 96 + \tilde{q}^1_2$, and $\tilde{f}_3 = \tilde{f}^1_3 = 92 + \tilde{q}^1_3$. If $\alpha = 3\%$, the schedule of Table 5 would be followed.

We can see that $\frac{max|p^j_h - p^j_i|}{max|p^j_g|}$ decreases steadily (with an occasional oscillation like it increases from 0.01826 to 0.02221 from row 6 to row 7). Another interesting aspect is that that $max|p^j_h - p^j_i|$ decreases at least every third term $\langle 1, 2, 3, 4, 5, 6, 9, 12, \cdots \rangle$.[10] Intuitively, we can see that whatever initial conditions we choose, the quantity $max|p^j_h - p^j_i|$ will tend to zero, as i increases. Furthermore the quantity $\frac{max|p^j_h - p^j_i|}{max|p^j_g|}$ will reach close to zero, and then start oscillating between three values (which are 0.01826, 0.02221, 0.01990 in our example).

Theorem E.1.1 *Whatever initial conditions we choose, the quantity* $max\{|p^j_h - p^j_i|\}$ *will tend to zero, as* j

[10] In other cases it the sequence might read $\langle \cdots, 7, 10, 13, \cdots \rangle$ or $\langle \cdots, 8, 11, 14 \cdots \rangle$.

increases. Moreover, the quantity $\frac{\max\{|p_h^j - p_i^j|\}}{\max p_g^j}$ *will reach close to zero, and then start oscillating between (at most) three values.*

Proof: If $p_1^j = p_2^j = p_3^j$ then the theorem is vacuously true. If two of the three values p_1^j, p_2^j, p_3^j are identical, then the analysis of § C.3 applies. If all three values are different then without loss of generality we can assume that $p_1^j > p_2^j > p_3^j$, and formally we can split the proof into five cases:

Case 1 : $p_1^j > p_2^j > p_3^j$**, and** $(1-\alpha)p_1^j > p_2^j > p_3^j$ $\Longrightarrow$ Optimal strategy will select $x_1^j = 1$ and $x_2^j = x_3^j = 0$. Thus, $\tilde{R}_P^j = \alpha \tilde{f}_1^j$. Furthermore, $\tilde{f}_1^{j+1} = (1-\alpha)\tilde{f}_1^j$, $\tilde{f}_2^{i+1} = \tilde{f}_2^j$ and $\tilde{f}_3^{i+1} = \tilde{f}_3^i$. Consequently, $p_1^{j+1} = (1-\alpha)p_1^j$, $p_2^{j+1} = p_2^j$ and $p_3^{j+1} = p_3^j$. We can show that the maximum difference between the stream levels decreases by showing that $max|p_h^{j+1} - p_i^{j+1}| \leq max|p_h^j - p_i^j|$ as follows: $max|p_h^{j+1} - p_i^{j+1}| = |p_1^{j+1} - p_3^{j+1}| = |(1-\alpha)p_1^j - p_3^j| = (1-\alpha)p_1^j - p_3^j$ [11] $\leq p_1^j - p_3^j = |p_1^j - p_3^j|$ $= max|p_h^j - p_i^j|$. In fact, $\frac{max|p_h^{j+1} - p_i^{j+1}|}{max|p_h^j - p_i^j|} = \frac{(1-\alpha)p_1^j - p_3^j}{p_1^j - p_3^j}$.

Case 2 : $p_1^j > p_2^j > p_3^j$**, and** $(1-\alpha)p_1^j = p_2^j > p_3^j$ $\Longrightarrow$ This reduces to a scenario similar to two-parameter stream model, which is already discussed in § C.3.

Case 3 $p_1^j > p_2^j > p_3^j$**, and** $p_2^j > (1-\alpha)p_1^j > p_3^j$ $\Longrightarrow$ In this case, we repeatedly alternate with $x_1^j = 1$, $x_2^{j+1} = 1$, $x_1^{j+2} = 1$, $x_2^{j+3} = 1$, ... until we reach Case 4 or 5.

Case 4 $p_1^j > p_2^j > p_3^j$**, and** $p_2^j > (1-\alpha)p_1^j = p_3^j$ $\Longrightarrow$ This reduces to a scenario similar to two-parameter stream model, which is already discussed in § C.3.

Case 5: $p_1^j > p_2^j > p_3^j$**, and** $p_2^j > p_3^j > (1-\alpha)p_1^j$ $\Longrightarrow$ Optimal strategy will select $x_1^j = 1$ and $x_2^j = x_3^j = 0$. Thus $\tilde{f}_1^{j+1} = (1-\alpha)\tilde{f}_1^j$, $\tilde{f}_2^{j+1} = \tilde{f}_2^j$ and $\tilde{f}_3^{j+1} = \tilde{f}_3^j$. Consequently, $p_1^{j+1} = (1-\alpha)p_1^j$, $p_2^{j+1} = p_2^j$ and $p_3^{j+1} = p_3^j$.
Now, since $p_2^{j+1} > p_3^{j+1}$, the second move selects $x_2^j = 1$, and $x_1^j = x_3^j = 0$, leaving $p_1^{j+2} = (1-\alpha)p_1^j$, $p_2^{j+2} = (1-\alpha)p_1^j$, and $p_3^{j+2} = p_3^j$.
The third move selects $x_3^j = 1$, and $x_1^j = x_2^j = 0$, leaving $p_1^{j+3} = (1-\alpha)p_1^j$, $p_2^{j+3} = (1-\alpha)p_1^j$, and $p_3^{j+3} = (1-\alpha)p_3^j$.
Consequently, we can show that $max|p_h^{j+3} - p_i^{j+3}| \leq max|p_h^j - p_i^j|$ because: $max|p_h^{j+3} - p_i^{j+3}| = p_1^{j+3} - p_3^{j+3} = (1-\alpha)p_1^j - (1-\alpha)p_3^j = (1-\alpha)(p_1^j - p_3^j) = max|p_h^j - p_i^j|$ In fact, $\frac{max|p_h^{j+1} - p_i^{j+1}|}{max|p_h^j - p_i^j|} = (1-\alpha)$.

The theorem follows. □

It is clear that the sequence $max\{|p_h^j - p_i^j|\}$ tends to 0 as j increases. It remains to be gauged how fast it approaches zero, i.e. find $N(\epsilon)$ such that $max\{|p_h^j - p_i^j|\} < \epsilon$ for $i > N(\epsilon)$. Without loss of generality, assume $p_1^1 > p_2^1 > p_3^1$. For $(1-\alpha)^n p_1^1 < p_3^1 < (1-\alpha)^{n-1} p_1^1$ and $(1-\alpha)^m p_2^1 < p_3^1 < (1-\alpha)^{m-1} p_2^1$, it would take $n + m = \frac{\log(p_1^1/p_3^1)}{\log(\frac{1}{1-\alpha})} + \frac{\log(p_2^1/p_3^1)}{\log(\frac{1}{1-\alpha})}$ moves. After the cross-over, $max|p_h^j - p_i^j|$ decreases by a factor of $(1-\alpha)$ every third iteration. This leads to following corollary.

Corollary E.1.1 $max|\{|p_h^j - p_i^j|\} < \epsilon_1$ *for* $j > 3\frac{\log(1/\epsilon)}{\log(\frac{1}{1-\alpha})} + \frac{\log(p_1^j/p_3^j)}{\log(\frac{1}{1-\alpha})} + \frac{\log(p_2^j/p_3^j)}{\log(\frac{1}{1-\alpha})}$, *where* $\epsilon_1 = \epsilon(max\{|p_h^a - p_i^a|\})$ *with a being the place where* p_B^n *and* p_A^n *"cross-over" for the first time.*

Corollary E.1.2 *If there is any* $C(k)$*-predictable disparity between the levels of two or more streams, the players would exploit it for their own benefit. Whatever initial conditions we choose, the difference between* $C(k)$*-predictable components of any two streams will tend to zero, with each move, until it reaches very close to zero. Consequently, the "inefficiencies" in the market are removed as the* $C(k)$*-predictability vanishes.*

The following points are worth observing:

1. There is a near linear decay of the p_1^j until it "crosses over" p_2^j.
2. There is a near linear decay alternating between p_1^j and p_2^j until they "cross over" p_3^j.
3. Decay of the maximum difference $max|p_h^j - p_i^j|$ is bounded above by a near linear monotonic function.
4. $C(k)$-unpredictable random variables retain their original distribution with a scaled mean and variance.

Even though, we assume that the $C(k)$-unpredictable components are insignificant compared to the $C(k)$-predictable components, $C(k)$-unpredictable components scale in the same proportion as the $C(k)$-predictable components in the same stream.

Once we have developed this basic formulation, we are interested in effects of the optimal strategy pursued by the player. There is considerable analysis that deals with risk sensitive issues (and the corresponding formula for standard deviation is stated in Appendix G), but we shall analyze the case where our players are risk apathetic, and are only concerned with maximizing their expected return.

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[11] since $(1-\alpha)p_1^j - p_3^j > 0$

Farmer	Stream 1 q_1	p_1	Stream 2 q_2	p_2	Stream 3 q_3	p_3	$max\lvert p_h^j - p_i^j\rvert$	$\frac{max\lvert p_h^j - p_i^j\rvert}{max\lvert p_g^j\rvert}$	x_1^j	x_2^j	x_3^j
1	$\tilde{Q}+$	100	$\tilde{Q}+$	96	$\tilde{Q}+$	92	8.00	0.0800	1.0	0.0	0.0
2	$0.9704\tilde{Q}+$	97.04	$\tilde{Q}+$	96	$\tilde{Q}+$	92	5.04	0.0520	0.6804	0.3196	0.0
3	$0.9508\tilde{Q}+$	95.08	$0.9905\tilde{Q}+$	95.08	$\tilde{Q}+$	92	3.08	0.0324	0.5	0.5	0.0
4	$0.9367\tilde{Q}+$	93.67	$0.9757\tilde{Q}+$	93.67	$\tilde{Q}+$	92	1.67	0.0178	0.5	0.5	0.0
5	$0.9227\tilde{Q}+$	92.27	$0.9611\tilde{Q}+$	92.27	$\tilde{Q}+$	92	0.27	0.0029	0.3659	0.3659	0.2682
6	$0.9126\tilde{Q}+$	91.26	$0.9506\tilde{Q}+$	91.26	$0.9920\tilde{Q}+$	91.26	0.00	0.0000	0.3333	0.3333	0.3333
7	$0.9035\tilde{Q}+$	90.35	$0.9411\tilde{Q}+$	90.35	$0.9821\tilde{Q}+$	90.35	0.00	0.0000	0.3333	0.3333	0.3333
8	...	...	...	...	...	...	...	...	...	...	...
9	...	...	...	...	...	...	...	...			

Table 6: Changes in Levels in Three-Parameter Stream Model with Dynamic Adjustment

E.2 Effect of Dynamic Adjustment

In the case of dynamic adjustment of stream water level to withdrawal, the return will be: $R_P^j = \sum_i (1 - e^{-\alpha_j x_i^j})\tilde{f}_i^j$, and the levels of stream for player $j+1$ can be calculated from the levels of player j by the formula $\tilde{f}_i^{j+1} = (e^{-\alpha_j x_i^j})\tilde{f}_i^j$.

We shall now analyze DAM$_1$ to build our intuition. DAM$_1$ assumes $\tilde{q}_i^j << p_i^j$ for all streams i, and $\tilde{q}_i^j << |p_k^j - p_l^j|$ for any three streams i, k, and l. Hence, jth farmer's optimal strategy to maximize R_P^i effectively means choosing $\langle x_i^j \rangle$ so as to maximize $max_{\langle x_i^j \rangle}[\sum_i (1 - e^{-\alpha_j x_i^j})\tilde{f}_i^j]$.

Suppose that initially $\tilde{f}_1 = \tilde{f}_1^1 = 100 + \tilde{q}_1^1$, $\tilde{f}_2 = \tilde{f}_2^1 = 96 + \tilde{q}_2^1$, and $\tilde{f}_3 = \tilde{f}_3^1 = 92 + \tilde{q}_3^1$. If $\alpha = 3\%$, the schedule of Table 6 would be followed.

The first move is obvious, and we choose $x_1^1 = 1.0$ (and $x_2^1 = 0.0$, $x_3^1 = 0.0$). The second move involves minimizing $100e^{-\alpha - \alpha x_1^2} + 96e^{-\alpha x_2^2} + 92e^{-\alpha x_3^2}$ subject to the constraint that $x_1^2 + x_2^2 + x_3^2 = 1.0$. This gives $x_1^2 = 0.6804$, $x_2^2 = 0.3196$, and $x_3^2 = 0.0$, and leaves $p_1^3 = p_2^3 = 95.08$, and $p_3^3 = 92$. The next two moves involve distributing resources equally between 1 and 2, until $p_1^5 = p_2^5 = 92.27$, and $p_3^5 = 92$.

The next move requires minimizing $p_1^5 e^{-\alpha x_1^5} + p_2^5 e^{-\alpha x_2^5} + p_3^5 e^{-\alpha x_3^5}$, with the constraint that $x_1^5 + x_2^5 + x_3^5 = 1$. By symmetry, $x_1^5 = x_2^5$, hence need to minimize the remaining level of the streams: $H(x_1^5, x_3^5, \lambda) = 2p_1^5 e^{-\alpha x_1^5} + p_3^5 e^{-\alpha x_3^5} + \lambda(2x_1^5 + p_3^5 - 1)$.

Consequently to find the minimum, we set the derivate of H with respect to x_1^5, x_3^5, and λ equal to 0: $\frac{dH}{dx_1^5} = -2\alpha p_1^5 e^{-\alpha x_1^5} + 2\lambda = 0$; $\frac{dH}{dx_3^5} = -\alpha p_3^5 e^{-\alpha x_3^5} + \lambda = 0$; $\frac{dH}{d\lambda} = 2x_1^5 + x_3^5 - 1 = 0$. Solving these equation, we get: $x_1^5 = x_2^5 = \frac{\alpha + \log(\frac{p_1^5}{p_3^5})}{3\alpha}$; $x_3^5 = \frac{\alpha - 2\log(\frac{p_1^5}{p_3^5})}{3\alpha}$.

For the values in the example: $x_1^5 = x_2^5 = 0.3659$, and $x_3^5 = 0.2682$. Subsequently, the predictable components of all streams are equal, hence the Player 6 onwards would optimally allocate $\frac{1}{3}$ of α to every stream. We can see that $\frac{max|p_h^j - p_i^j|}{max|p_g^j|}$ diminishes steadily at a near linear rate until it is equal to 0. Similarly, the difference $max|p_h^j - p_i^j|$ approaches 0 at a near-linear rate.

Theorem E.2.1 *Whatever initial conditions we choose, the quantity* $max\{|p_h^j - p_i^j|\}$ *will approach zero, as* j *increases, and will be zero for all but a finite number of* j*'s. Moreover, the quantity* $\frac{max\{|p_h^j - p_i^j|\}}{max p_g^j}$ *will also reach zero.*

Proof: If $p_1^j = p_2^j = p_3^j$ then the theorem is true vacuously. If two of the three values p_1^j, p_2^j, p_3^j are identical, then the analysis of § C.4 applies. If all three values are different then without loss of generality we can assume that $p_1^j > p_2^j > p_3^j$, and formally we can split the proof into three cases:

Case 1 : $p_1^j > p_2^j > p_3^j$, **and** $e^{-\alpha} p_1^j > p_2^j > p_3^j \implies$

Optimal strategy will select $x_1^j = 1.0$ and $x_2^j = x_3^j = 0.0$. Thus $\tilde{f}_1^{j+1} = e^{-\alpha}\tilde{f}_1^j$, $\tilde{f}_2^{j+1} =, \tilde{f}_2^j$ and $\tilde{f}_3^{j+1} = \tilde{f}_3^j$. Consequently, $p_1^{j+1} = e^{-\alpha}p_1^j$, $p_2^{j+1} = p_2^j$ and $p_3^{j+1} = p_3^j$. We can show that $max|p_h^{j+1} - p_i^{j+1}| \leq max|p_h^j - p_i^j|$ because:

$max|p_h^{j+1} - p_i^{j+1}| = |p_1^{j+1} - p_3^{j+1}| = |e^{-\alpha}p_1^j - p_3^j| = e^{-\alpha}p_1^j - p_3^j \leq p_1^j - p_3^j = |p_1^j - p_3^j| = max|p_h^j - p_i^j|$. In fact, $\frac{max|p_h^{j+1} - p_i^{j+1}|}{max|p_h^j - p_i^j|} = \frac{e^{-\alpha}p_1^j - p_3^j}{p_1^j - p_3^j}$.

Case 2 $p_1^j > p_2^j > p_3^j$, **and** $p_2^j > e^{-\alpha}p_1^j > p_3^j \implies$

F A Network Representation

F.1 Basic Network Nodes: MERGER, SPLIT, and RETURNS

In this subsection, we add present a generalized model to emulate most features of investment markets. We have already analyzed the concepts of return on investment in a multiparameter capital market. However, we need to emulate mergers and splits into our model. We now describe and analyze the most versatile model of this paper, which uses flow of time-dependent streams in a network to emulate capital markets.

Our network model is based on time-dependent stream flows in a network with functional nodes, which regulate the flow according to the specified rules. There are three main nodes RETURN, MERGER, and SPLIT, and these are illustrated in Figure 7. We describe each node in turn, and then explain a simple network model using these nodes.

MERGER NODE: The MERGER nodes combines the flow of two streams with levels f_1 and f_2 to produce a new stream with a level of f, such that $f = f_1 + f_2$.

This is reminiscent of a corporate merger, where two corporations, two products, or two markets merge to form one corporation, one product, or one market, respectively. For example, Kansas wheat merges with Nebraska wheat to be sold through the same distributor, or a beef supplier and a bakery decide to merge and sell hamburgers.

We must note that the sum of the total incoming flows to the node is equal to the total outgoing flow, hence $f = f_1 + f_2$. The particulars of each merger might involve complicated computations, but we can easily use the formula specifying the particulars of a particular merger. As illustrated in Figure 7(a), MERGER node is denoted by an obtuse triangle, with the two incoming streams (securities) incident onto the largest side of the triangle, and the resultant stream (security) originating from the obtuse angle of the triangle.

SPLIT NODE : The SPLIT node bifurcates one stream into two substreams, with each stream getting a fraction of the parent streams contents, as specified by the split parameter.

The SPLIT node represents a corporation (or a product, or a market) splitting into two different corporations (or products, or markets, respectively). For example, Kansas wheat is distributed between two distributors, one for for eastern United States, and one for western United States, or a poultry supplier splitting into two suppliers, one for white meat and the other for dark meat.

The incoming flow to SPLIT node is equal to the total outgoing flow. Consequently, $f_1 + f_2 = f$. As illustrated in Figure 7(b), SPLIT node is denoted by an obtuse triangle, with the incoming stream (security) incident onto the only obtuse angle, and the resultant sub-streams (sub-securities) originating from the largest side of the triangle.

RETURN NODE : The RETURN node taps the amount of withdrawal a farmer is making for from the market.

The RETURN node denotes the return on an investors investment or the resultant profit from allocation of resources/rights. For example, if a farmer decides to withdraw a fraction γ water from a stream with level f, his return would be $R = \gamma f$ for Initial Level model, and $R = (1 - e^{-\gamma})f$ for Dynamic Adjustment model. Subsequently, the flow left in the stream would be $f' = (1 - \gamma)f$ for Initial Level model, and $f' = e^{-\gamma}f$ for Dynamic Adjustment model. As illustrated in Figure 7(c), RETURN node is denoted by a circle which encapsules the return parameter, and a rectangle to indicate the farm receiving the the water withdrawn from the stream.

F.2 Efficiency in the Network Model

The stream model and security markets can be enhanced to accommodate splits, mergers, and dividends, as described in the previous subsection. Once, we have chosen to represent a security market by the representation above, we can abstract all information about it, and depict it by a simple network flow diagram.

This subsection shows that analysis of this flow diagram is much simpler than we might expect: The simplicity is a direct benefit of the rigorous tools developed earlier. The given network is cyclic in the sense that it contains a loop in which resources can cycle forever. However, we can remove these cycles by unfolding the network over time. Such unfolding would yield a blow up to an exponential blow up in space every time we unfold. However, it would enable us to analyze any cyclic network within some finite time.

Once we are presented with a acyclic network, it is a matter of applying the following three rules to complete the analysis:

Combine Rule : *Whenever we see k (two or more) streams merging into one stream, we combine the flows of all the streams to compute the resultant combined flow:* $\sum_i p_i^{in} = p_i^{out}$.

We must point out that the simple addition rule is replaced by appropriate formulae depending on the the merger arrangements. For the purpose of this paper, we shall assume that a merger involves a simple addition because the particular details of the formula that quantitatively defines a merger are peripheral to our formalization attempts.

Bifurcate Rule : *Whenever we see one stream splitting into k (two or more) streams, we branch the flows according to the split parameters:* $p_i^{out} = \alpha_i p^{in}$, where $(\sum_i \alpha_i) = 1$

Once again, the simple co-efficient rule is replaced by appropriate formulae depending on the the split arrangements. We shall assume that a split involves a simple co-efficient formula because the particular details of the formula defining a split are peripheral to our main objective.

Clamp Rule : *Whenever we see k (two or more) streams accessible to some farm j, we equate all their apparent expected flows according to Theorem E.3.1 of § E.3 (which assumes short-selling is permitted):* $p_i^{out} = e^{-\frac{1}{k}\alpha}(\prod_{i=1}^{k} p_i^{in})^{1/k}$.

The RETURN node is central to the entire paper, since it captures the essence of the most frequent occurrence in the investment market: An investor deriving a return from his investment. We first developed the prelude to Clamp Rule in § D.3, and this was later extended in § E.3. The Clamp Rule is valid even if short-selling is disallowed on limited in anyway, except it takes more players without short-selling opportunity to accomplish what one player accomplished with unrestricted short-selling.

The effect of Bifurcate and Combine rules is very simple and straight-forward in their effect and analysis. However, the Clamp rule is the most important to our analysis, and most intricate to analyze. As shown above, it is apparent that Clamp rule has an effect similar to voltage clamps in electrical networks: It forces all associated stream flows to be identical. Once that happens, there is no disparity in $\mathcal{C}(k)$-predictability left for subsequent investors to exploit. Equipped with these three nodes, we can abstract the essence out of any network model.

G Standard Deviations for Risk Analysis in Two-parameter One-period Stream Model

We have analyzed expected return for risk apathetic players in this paper. There is considerable analysis that deals with risk aversion issue and special nuances of the two-parameter model by Markowitz [12, 13], Merton [14], Sharpe [18], Lintner [11], Brennan [6], and Black [2]. The risk-return can be analyzed using the following formulae for standard deviations in returns for our two-parameter one-period stream model.

For the two-parameter one-period stream initial value model, the standard deviation is:

$$\sigma[R_P] = \alpha\sqrt{(x^2\sigma^2[\tilde{f}_A] + (1-x)^2\sigma^2[\tilde{f}_B] + 2x(1-x)corr(\tilde{f}_A,\tilde{f}_B)\sigma[\tilde{f}_A]\sigma[\tilde{f}_B])}$$

$$= \alpha\sqrt{(x^2\sigma^2[\tilde{q}_A] + (1-x)^2\sigma^2[\tilde{q}_B] + 2x(1-x)corr(\tilde{q}_A,\tilde{q}_B)\sigma[\tilde{q}_A]\sigma[\tilde{q}_B])},$$

where $corr(\tilde{X},\tilde{Y}) = \frac{cov(\tilde{X},\tilde{Y})}{\sigma[\tilde{X}]\sigma[\tilde{Y}]}$.

For the two-parameter one-period stream model with dynamic adjustment, the standard deviation $\sigma[R_P] =$

$$= \sqrt{((1-e^{-\alpha x})^2\sigma^2[\tilde{f}_A] + (1-e^{-\alpha(1-x)})^2\sigma^2[\tilde{f}_B] + 2(1-e^{-\alpha x})(1-e^{-\alpha(1-x)})corr(\tilde{f}_A,\tilde{f}_B)\sigma[\tilde{f}_A]\sigma[\tilde{f}_B])}$$

$$= \sqrt{((1-e^{-\alpha x})^2\sigma^2[\tilde{q}_A] + (1-e^{-\alpha(1-x)})^2\sigma^2[\tilde{q}_B] + 2(1-e^{-\alpha x})(1-e^{-\alpha(1-x)})corr(\tilde{q}_A,\tilde{q}_B)\sigma[\tilde{q}_A]\sigma[\tilde{q}_B])}.$$

For k-parameter stream model, the standard deviation governing the risk of Player j for Portfolio P is denoted by R_P^j and computed by the following formula:

$$\sigma[R_P^j] = \alpha\sqrt{\sum_i x_i^j\sigma^2[\tilde{f}_i] + \sum_i\sum_k x_i^j x_k^j corr(\tilde{f}_i,\tilde{f}_k)} \quad = \alpha\sqrt{\sum_i x_i^j\sigma^2[\tilde{q}_i] + \sum_i\sum_k x_i^j x_k^j corr(\tilde{q}_i,\tilde{q}_k)}$$