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A Survey on Advances in the Theory of Computational Robotics

John H. Reif
Aiken Computation Laboratory
Division of Applied Sciences
Harvard University
Cambridge, Massachusetts 02138

Abstract

This paper describes work on the computational complexity of various movement planning problems relevant to robotics. This paper is intended only as a survey of previous and current work in this area. The generalized mover's problem is to plan a sequence of movements of linked polyhedra through 3-dimensional Euclidean space, avoiding contact with a fixed set of polyhedra obstacles. We discuss our and other researchers' work showing generalized mover's problems are polynomial space hard. These results provide strong evidence that robot movement planning is computationally intractable, i.e., any algorithm requires time growing exponentially with the number of degrees of freedom. We also briefly discuss the computational complexity of four other quite different types of movement problems: (1) movement planning in the presence of friction, (2) minimal movement planning, (3) dynamic movement planning with moving obstacles and (4) adaptive movement planning problems.

1. Introduction: The Mover's Problem

The classical *mover's problem* in d-dimensional Euclidean space is:
Input: (O, P, p_I, p_F) where O is a set of polyhedral *obstacles* fixed in Euclidean space and P is a rigid polyhedron with distinguished initial position p_I and final position p_F. The inputs are assumed to be specified by systems of rational linear inequalities.
Problem: Can P be moved by a sequence of translations and rotations for p_I to p_F without contacting any obstacle in O?

For example, P might be a sofa[1] which we wish to move through a room crowded with obstacles. Figure 1 gives a simple example of a two-dimensional mover's problem.

The mover's problem may be *generalized* to allow P (the object to be moved) to consist of multiple polyhedra freely linked together at various distinguished vertices. (A typical example is a robot arm with multiple joints.) Again, the input is specified by systems of rational linear inequalities.

The paper is organized as follows: Section 2 concerns lower bounds for generalized mover's problems in 2D and 3D. Section 3 concerns efficient solution of restricted mover's problems. Section 4 concludes the paper with discussion of further problems in computational robotics.

2. Lower Bounds for Generalized Mover's Problems

In [1] (also appearing in [2]) we proved that the generalized mover's problem in three dimensions is polynomial space hard. That is, we proved that the generalized mover's problem is at least as hard as any computational problem requiring polynomial space. (Polynomial space problems are at least as hard as the well known NP problems; see [3].)

[1]The author first realized the nontrivial mathematical nature of this problem when he had to plan the physical movement of an antique sofa from Rochester to C[illegible]dge.

This was the first paper investigating the inherent computational complexity of a robotics problem in Computational Geometry. Our proof technique is to use the degrees of freedom of P to encode the configuration of a polynomial space bounded Turing machine M, and to design obstacles which forced the movement of P to simulate the computation of M.

This work was originally motivated by application to robotics: the author felt it was important to examine *computational complexity issues in robots* given the recent development of mechanical devices autonomously controlled by micro and minicomputers, and the swiftly increasing computational power of these controllers. However, it took a number of years before computational complexity issues in robotics became of more general interest. Recently there have been a flurry of papers in the now emerging area which we might term *Computational Robotics.*

Recent investigations in lower bounds have provided some quite ingenious lower bound constructions for restricted cases of the generalized mover's problem. For example, [4] showed that the generalized mover's problem in three dimensions is also polynomial space hard, and [5] showed that the problem of moving a collection of disconnected polyhedra in a two-dimensional maze is polynomial space hard. The problem of moving a collection of disks in two dimensions is known to be NP-hard [6], but it remains open to show this problem polynomial space hard.

3. Upper Bounds for Mover's Problems

Our lower bounds for the generalized mover's problem provided evidence that time bounds for algorithms for movement planning must grow exponentially with the number of degrees of freedom. We next give a brief description of known algorithms for mover's problems. In our original paper [1] we also sketched a method for efficient solution of the classic mover's problem where P, the object to be moved, is rigid. In spite of considerable work on this problem by workers in the robotics fields and in artificial intelligence (for example [7-11]), no algorithm guaranteed to run in polynomial time had previously appeared. Our approach was to transform a classic mover's problem (O, P, p_I, p_F) of size n in d dimensions to an apparently simpler mover's problem (O', P', p'_I, p'_F) of dimensions d', where P' is a single part and d' is the number of degrees of freedom of movement in the original problem. The transformed problem is thus to find a path in d'-dimensional space avoiding the transformed obstacles O. The fundamental difficulty is that the induced obstacles may be nonlinear constraints. (In [11], Lozano-Pérez and Wesley did not construct O', but instead approximated the induced obstacles O' by linear constraints. Unfortunately, an exponential number of linear constraints were required to approximate even a quadratic constraint within accuracy 2^{-n}. Thus their method required exponential time (i.e., 2^{cn} time for some $c > 0$) even if the original mover's problem was two-dimensional.)

Example. Consider a classical mover's problem (O, P, p_I, p_F) restricted to *dimension* $d = 2$, with the obstacles O consisting of a set of line segments and P a single polygon. A *position* of P can be specified by a triple (x, y, θ) where (x, y) are the Cartesian coordinates of some fixed vertex of P and θ is the angle of rotation around this vertex. We define a mapping f from the position of P to 3-space. Let $f(x, y, \theta) = (x', y', z')$ where $y = z'$, $tan(\theta) = x'/y'$, and $x = (x')^2 + (y')^2 - \alpha$, for some sufficiently large constant $\alpha \geq 0$. (α may be taken as the diameter of a circle enclosing P.) See Figure 3.

In this case, we define a 1-*contact set* to be a maximal set of positions of P where a vertex of P contacts a line segment of O, or a vertex of O contacts a line segment of P. (See Figure 4.) The transformed obstacles O' are the union of these 1-contact sets. Thus each obstacle in O' is a quadratic surface patch which may be easily constructed from the input, there are at most $O(|O||P|)$ such obstacles and their $O(|O|^2|P|^2)$ intersections can easily be computed within accuracy $2^{n^{-c}}$ for any $c > 0$, by known polynomial time procedures [12] for intersection of quadratic surface patches. Hence in this simple example the connected regions bounded by O' can be explicitly constructed in polynomial time within accuracy 2^{-n^c} which is sufficient for solution of this mover's problem.

In the case of a classical mover's problem (O, P, p_I, p_F) of dimension $d = 3$, the transformed problem (O', P', p'_I, p'_F) has dimension $d' = 6$. In this case we define a 1-contact set to be a maximal set of positions of P where an edge of P contacts a face of O or an edge of O contacts a face of P. Again, the 1-contact sets are constant degree polynomial. The transformed obstacles O' are the union of the 1-contact sets. The connected regions defined by O' can again be explicitly constructed by intersecting these constraints. In [1], we briefly suggested a method for this construction, but the full credit should be given to [13] who later gave a complete detailed description of a method for explicit construction of such a transformed mover's problem in 3 dimensions in polynomial time. (In [14] O'Dunlaing et al. further improved this construction by observing that movement of P can be restricted to be equidistant from the obstacles.)

This approach was extended in [15] to solve any generalized mover's problem of input size n with d' degrees of freedom in time $n^{2^{O(d')}}$. They make use of the algebraic decomposition of [16] (previously used to decide formulas of the theory of real closed fields) to construct the connected regions bounded by O'. Note that their upper bounds grow doubly exponentially with d', whereas our polynomial space lower bounds suggest only single exponential time growth with d'. It remains a challenging problem to close the gap between those lower and upper bounds for generalized mover's problems. Further progress will likely depend on improvements to decision algorithms for the theory of real closed fields; recently Ben-Or et al. gave a single exponential space decision algorithm [17].

4. Further Problems in Computational Robotics

There are some very challenging problems remaining in the field of Computational Robotics beyond the complexity of the mover's problems and some recent progress.

(1) Frictional Movement. The problem here is to plan movement for (O, P, p_I, p_F) in the case contact is allowed in the presence of friction between surfaces. In [18], Rajan and Schwartz give the first known decision algorithm in the case that O is a cylindrical hole and P is a peg. In [19], Miller and Reif prove undecidability of planning frictional movement. What natural subclass of frictional movement problems is decidable?

(2) Minimal Movement. The problem is, given a set of k polygonal obstacles in d space defined by a total of n linear constraints, and points p_I, p_F find a minimal length path from p_I to p_F avoiding the obstacle O. [20] gives a $O(n \log n)$ algorithm in the case $d = 2$ and $k = 1$. [21] give a $2^{2^{O(n)}}$ algorithm for $d = 3$. Recently, Reif and Storer [22] gave a $O(nk \log n)$ algorithm for $d = 2$ and $n^{k^{O(1)}}$ time and $n^{O(\log k)}$ space algorithms for $d = 3$. Is there a $n^{O(1)}$ algorithm for $d = 3$?

(3) Dynamic Movement. The problem is to plan the movement of a polygon in d dimensions with bounded velocity modulus between points p_I and p_F, so as to avoid contact with a set O of k polygonal obstacles (defined by a total of n linear constraints) moving with fixed, known velocity. [22] give the first known investigation of the computational complexity of planning dynamic movement. They show that the problem of planning dynamic movement of a single ($k = 1$) disk P in $d = 3$ dimensions is polynomial space hard. (This result is somewhat surprising, since P in this case has only 3 degrees of freedom. Our key new idea is to use time to encode a configuration of a polynomial space bounded Turing machine.) Is this problem polynomial space hard for dimension $d = 2$?

Asteroid avoidance problems are a natural subclass of dynamic mover's problems where each obstacle is convex and does not rotate. In [22] Reif and Sharir give a polynomial time algorithm for dimension $d = 2$ with a bounded number $k = O(1)$ of obstacles and give $2^{n^{O(1)}}$ time and $n^{O(\log n)}$ space algorithms for dimension $d = 3$ with an unbounded number k of obstacles. Is the asteroid avoidance problem polynomial in the case $d = 3$?

(4) Adaptive Movement Planning. The problem is to do dynamic movement planning in the case where the obstacles make unpredicted movements in real time. This problem requires some sort of adaptive response to the changes of obstacles' trajectories, and appears considerably more difficult than the dynamic movement problem where the obstacles are assumed

to make predictable movements. Although no previous work has been done in this area, it seems to be of central importance.

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