The λ-Medial Axis

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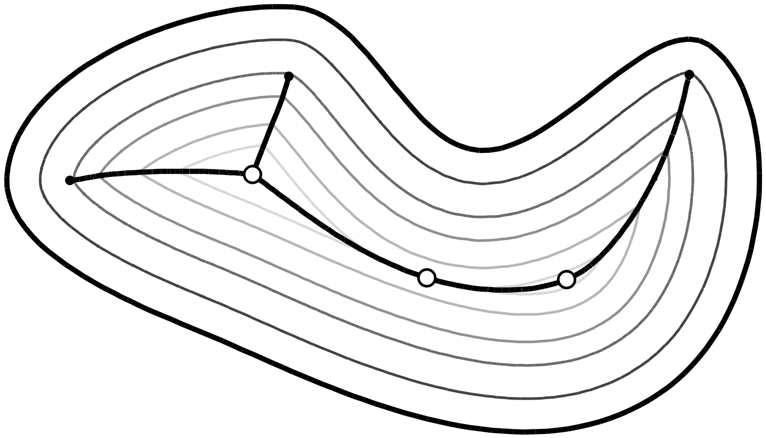
Medial Axis and Skeleton

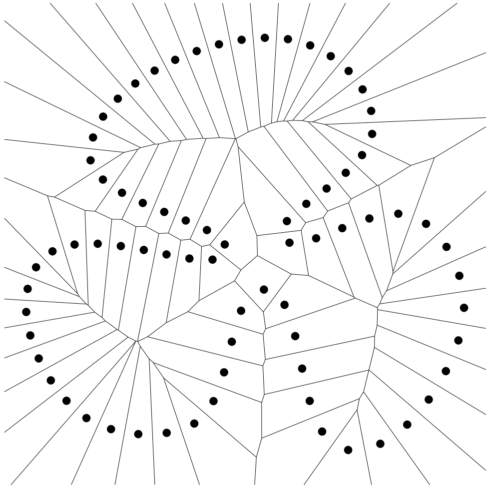
Given an open set $\mathcal{O} \subseteq \mathbb{R}^d$,

- skeleton: centers of all maximal (wrt inclusion) balls contained in $\mathcal{O}$
- medial axis: points with at least 2 closest points on $\partial\mathcal{O}$

Easy to check:

$$\text{medial axis} \quad \subseteq \quad \text{skeleton} \quad \subseteq \quad \text{closure of medial axis}$$





Topology of Medial Axis

Previously known:

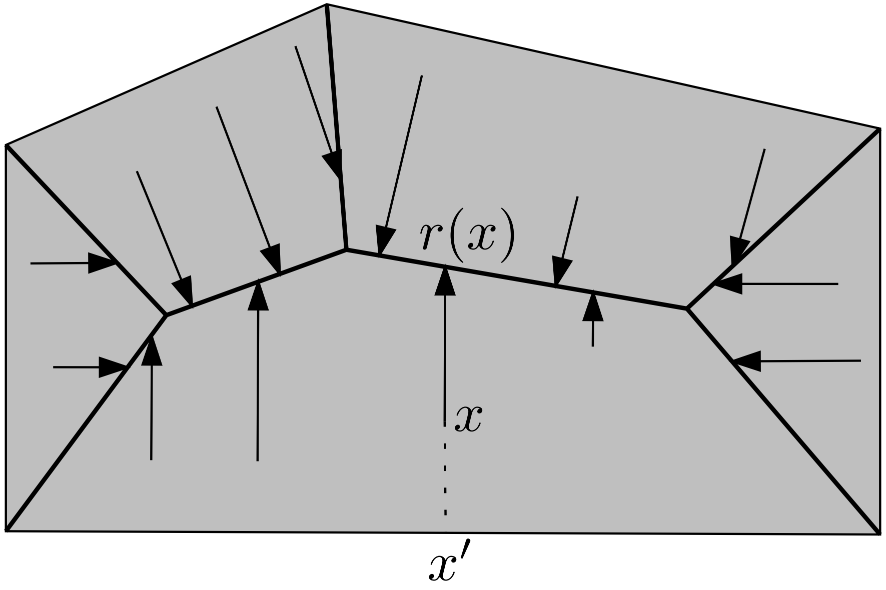
- [Matheron'88] Closure of MA of a connected open set is connected.
- [Wolter'93] Any open set in $\mathbb{R}^2$ with piece-wise C^2 boundary is homotopy equivalent to its MA.
- [Wolter'93] Any open set in $\mathbb{R}^n$ with C^2 boundary is homotopy equivalent to its MA.
- [Choi, Choi, Moon'97] MA of an open set in $\mathbb{R}^2$ with piece-wise analytic boundary is made of a finite number of curves.
- [Chazal, Soufflet'04] Same holds if boundary is piece-wise subanalytic in any dimension.

When boundary is C^2 ...

The proof of [Wolter'93] for case with smooth boundary defines a deformation retract $r : \mathcal{O} \to \mathcal{O}$

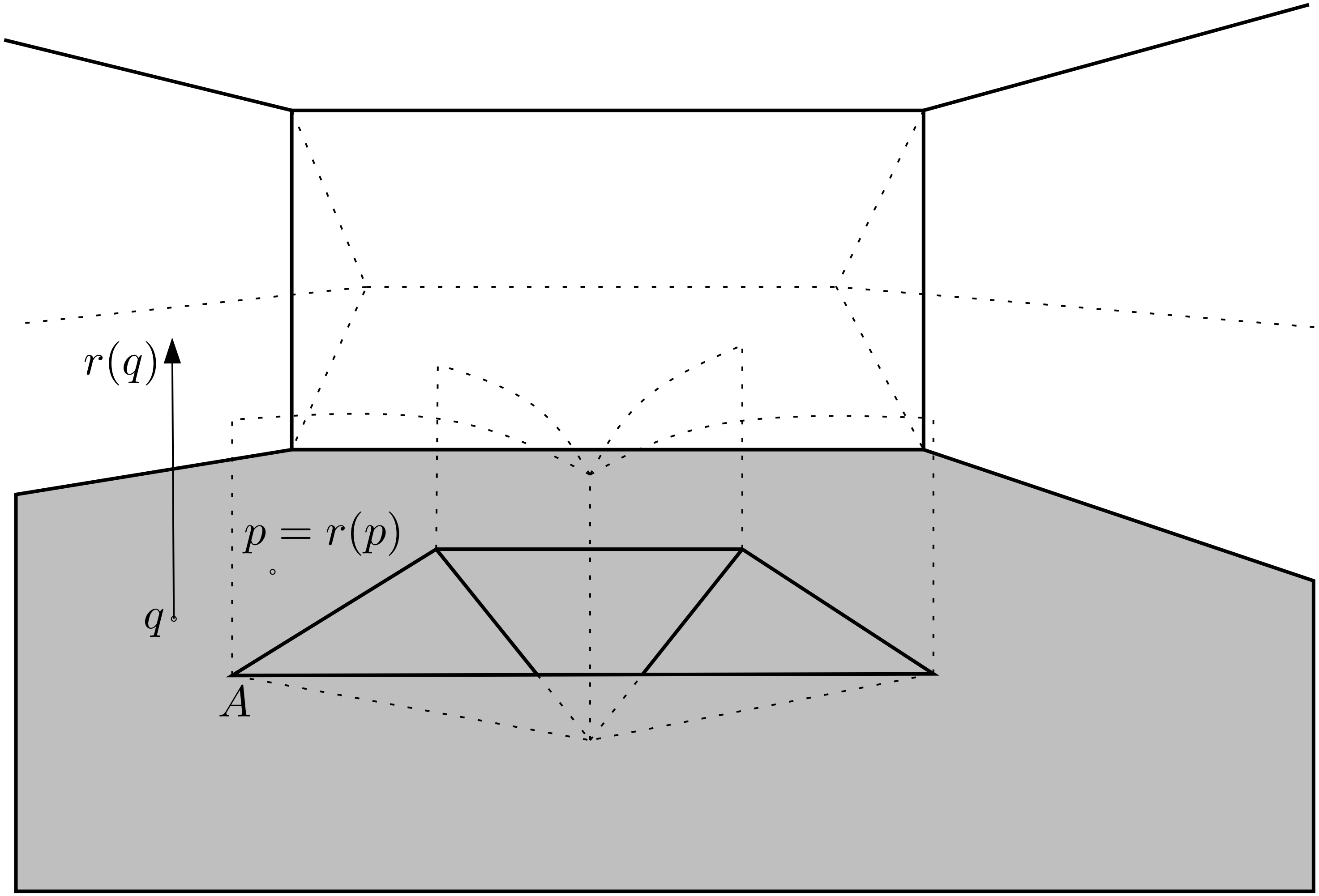
$$r(x) = \begin{cases} \text{First point of } \overline{\mathcal{M}} \text{ in direction } x'x & \text{if } x \notin \overline{\mathcal{M}} \\ x & \text{if } x \in \overline{\mathcal{M}} \end{cases}$$

where x' is the (unique) closest point in $\partial\mathcal{O}$ to x.



Fails when boundary is not C^2 ...

Room with a pyramid depression in the floor.



The idea for the general case

- The deformation retract $r : \mathcal{O} \rightarrow \mathcal{O}$ moves every point $x \in \mathcal{O} \setminus \mathcal{M}$ in the direction gradient of distance to ∂O at x.
- Gradient is not defined for points in $\mathcal{M}$
- But **steepest ascent** is still meaningful!

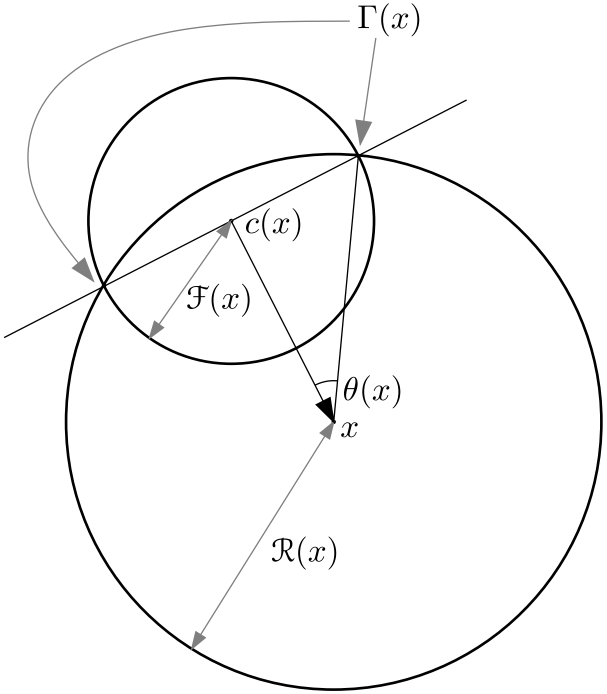
The idea is to replace gradient with the direction of steepest ascent and look at the resulting deformation.

Formally...

Let $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$

For a point $x \in \mathcal{O}$,

$$\begin{aligned} \mathcal{R}(x) &= d(x, \mathcal{O}^c) \\ \Gamma(x) &= \{y \in \mathcal{O}^c : d(x,y) = \mathcal{R}(x)\} \\ \mathcal{F}(x) &= \inf\{r : \exists y \in \mathbb{R}^n, \Gamma(x) \subset B(y,r)\} \\ \mathcal{O}_\rho &= \{x \in \mathcal{O} : \mathcal{R}(x) > \rho\} \end{aligned}$$



The **medial axis** is

$$\begin{aligned} \mathcal{M} &= \{x \in \mathcal{O} : |\Gamma(x)| > 1\} \\ &= \{x \in \mathcal{O} : \mathcal{F}(x) > 0\} \\ &= \{x \in \mathcal{O} : \|x - c(x)\| < \mathcal{R}(x)\} \end{aligned}$$

Flow Vector Field

For any $x \in \mathcal{O}$ define

$$\nabla(x) = \frac{x - c(x)}{\mathcal{R}(x)}$$

- ∇ extends the **gradient**.
- We have the identity

  $$\nabla(x)^2 = 1 - \frac{\mathcal{F}(x)^2}{\mathcal{R}(x)^2}$$

  So,

  $$\mathcal{M} = \left\{x \in \mathcal{O} : \|\nabla(x)\| < 1\right\}$$

- **Critical points** of ∇ are x for which

  $$\nabla(x) = 0,$$

  or identically x satisfying $x \in \text{conv}\ \Gamma(x)$

Integrating the Vector Field ∇

Consider the differential equation.

$$\begin{aligned} C(0,x) &= x \\ \frac{d}{dt}C(t,x) &= \nabla(C(t,x)) \end{aligned}$$

- It is not clear if this equation has a (unique) solution.
- ∇ is not continuous, let alone Lipschitz $\Rightarrow$ Picard's thm doesn't work.

Picard's Theorem. If $f(t,x)$ is continuously differentiable on a rectangle $[a,b] \times [c,d]$ in t,x-plane. The differential equation

$$\frac{dx}{dt} = f(t,x)$$

has some differentiable solution $u(t)$. Furthermore, if $f(t,x)$ is Lipschitz wrt x, then the solution is unique.

The Tools

h-Sampling

For a real interval $[a, b]$, the sequence

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

is an **h-sampling** if

$$t_i - t_{i-1} \leq h,$$

for all $i = 1, \ldots, n$.

Euler Schemes

Given vector map $V : X \to \mathbb{R}^n$ and real numbers

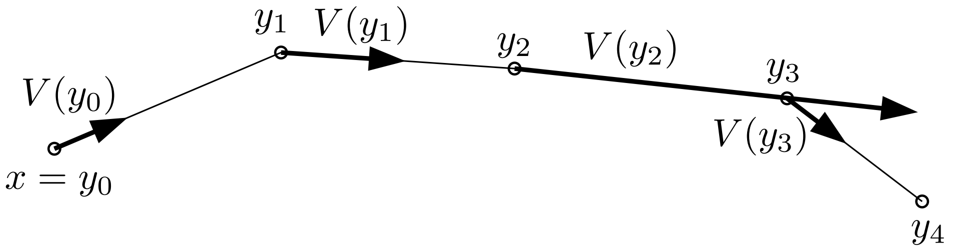
$$0 = t_0 < t_1 < \ldots < t_n = T,$$

fix $x \in X$, and define

$$\begin{aligned} y_0 &= x \\ y_{i+1} &= y_i + (t_{i+1} - t_i)V(y_i). \end{aligned}$$

Then the **euler scheme** $E^x_{t_0,\ldots,t_n} : \mathbb{R} \to \mathbb{R}^n$, is defined for $t \in [t_i, t_{i+1}]$ as:

$$E^x_{t_0,\ldots,t_n}(t) = y_i + (t - t_i)V(y_i).$$



Definition of flow $\mathcal{C}$

Fix $x \in \mathcal{O}$.

- Let $h_1, h_2, \ldots$ be a sequence converging to 0
- For each $i \in N$, let E_i^x be a Euler Scheme starting at x based on an h_i sampling of $\mathbb{R}$.

Then $\{E_i^x\}_{i\in\mathbb{N}}$ uniformly converges to a unique map

$$E^x(t) : \mathbb{R} \to \mathbb{R}^n$$

Letting x vary over $\mathcal{O}$, we get a map (**flow**)

$$\mathcal{C} : \mathbb{R} \times \mathcal{O} \to \mathcal{O}$$
$$\mathcal{C}(t, x) = E^x(t)$$

Global Existence of $\mathcal{C}$

Theorem. The Euler Schemes converge to a map

$$\mathcal{C} : \mathbb{R}^+ \times \mathcal{O} \to \mathcal{O}$$

which is:

- continuous
- 1-Lipschitz wrt real variable t

and its restriction to $\mathbb{R}^+ \times \mathcal{O}_\rho$ is Lipschitz wrt both variables, ie if $x, y \in \mathcal{O}_\rho$,

$$\|\mathcal{C}(t,x) - \mathcal{C}(t',y)\| \leq |t - t'| + \|x - y\| e^{\frac{1}{\rho} \min\{t,t'\}}$$

Some Useful Facts

In a series of tedious lemmas, Lieutier proves:

- The map $t \mapsto \nabla(\mathcal{C}(t, x))$ is the **right derivative** of $t \mapsto \mathcal{C}(t, x)$.
- The map $t \mapsto \nabla(\mathcal{C}(t, x))^2$ is the **right derivative** of $t \mapsto \mathcal{R}(\mathcal{C}(t, x))$.
- From the Lipschitzness of $t \mapsto \mathcal{C}(t, x)$ and $t \mapsto \mathcal{R}(\mathcal{C}(t, x))$, it is implied that

$$\begin{aligned} \mathcal{C}(t, x) &= x + \int_0^t \nabla(\mathcal{C}(\tau, x))d\tau \\ \mathcal{R}(\mathcal{C}(t, x)) &= \mathcal{R}(x) + \int_0^t \nabla(\mathcal{C}(\tau, x))^2 d\tau \end{aligned}$$

Also. . .

- The curve image of $t \mapsto \mathcal{C}(t,x)$ is rectifiable and its arc length as a function of t is

$$s(t) = \int_0^t \|\nabla(\mathcal{C}(\tau,x))\| d\tau$$

- If $s \mapsto t(s)$ is the inverse map,

$$\mathcal{R}(\mathcal{C}(t(s),x)) = \mathcal{R}(x) + \int_0^s \|\nabla(\mathcal{C}(t(\sigma),x))\| d\sigma$$

- The map $t \mapsto \mathcal{F}(\mathcal{C}(t,x))$ is increasing.
- The map $t \mapsto \mathcal{F}(\mathcal{C}(t,x))$ is upper-semi-continuous, i.e.

$$\forall \epsilon \in \mathbb{R}, \quad \{x \in \mathcal{O}, \mathcal{F}(x) < \epsilon\}$$

is open.

Using the Tools

A Criterion for Homotopy Equivalence

Proposition. If $Y \subset X$ and there exists a continuous map

$$H : [0,1] \times X \to Y,$$

such that

1. $\forall x \in X$, $H(0,x) = x$
2. $\forall x \in X$, $H(1,x) \in Y$
3. $\forall y \in Y, \forall t \in [0,1]$, $H(t,y) \in Y$

then, X and Y are **homotopy equivalent**.

If we replace 3. with

3′. $\forall y \in Y, \forall t \in [0,1]$, $H(t,y) = y$

then H defines a deformation retract of X towards Y.

Homotopy Type of Medial Axis

Theorem. If $\mathcal{O} \subset \mathbb{R}^n$ is open and bounded, then $\mathcal{O} \simeq \mathcal{M}$.

Proof. Let D be an upper bound for diameter of $\mathcal{O}$. Consider the map $H : [0,1] \times \mathcal{O} \to \mathcal{M}$ given by

$$H(t,x) = \mathcal{C}(Dt,x).$$

- since $t \mapsto \mathcal{F}(\mathcal{C}(t,x))$ is increasing, flow does not leave $\mathcal{M}$
- If $H(1,x) \notin \mathcal{M}$, then $\mathcal{F}(t,x) = 0$ for $t \in [0,1]$.

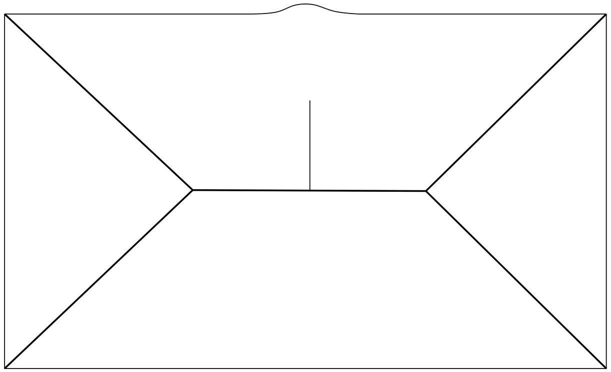
$$\nabla(x)^2 = 1 - \frac{\mathcal{F}(x)^2}{\mathcal{R}(x)^2} \qquad \Longrightarrow \qquad \nabla(\mathcal{C}(Dt,x)) = 1$$

Thus using $\mathcal{R}(\mathcal{C}(t,x)) = \mathcal{R}(x) + \int_0^t \nabla(\mathcal{C}(\tau,x))^2 d\tau$:

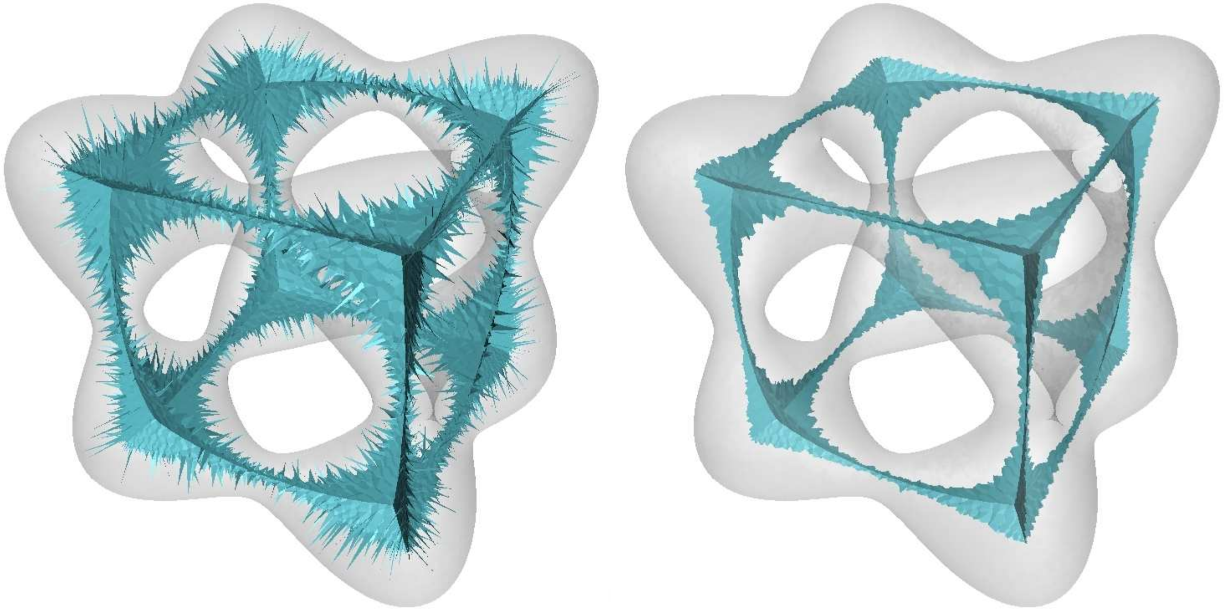
$$\mathcal{R}(\mathcal{C}(Dt,x)) \geq \mathcal{R}(x) + \int_0^D dt \geq D$$

Instability of Medial Axis

- $\mathcal{M}(\mathcal{O})$ is very sensitive to perturbations on $\partial\mathcal{O}$.



- In practice, presence of noise makes $\mathcal{M}(\mathcal{O})$ very hard to approximate.



A Suitable Substitute For MA

Things we may wish to have in a replacement for $\mathcal{M}$:

- Converges to $\mathcal{M}$ when some parameter moves towards a limit.
- Be stable (small Hausdorff distance between objects results small Hausdorff distance between (substitute) medial axes.
- Represents the object: carries the homotopy type.

A Bad Choice: Filtering out points x that have small $\theta(x)$.

A Good Choice

The λ-**medial axis** is defined as

$$\mathcal{M}_\lambda = \{x \in O : \mathcal{F}(x) \geq \lambda\}.$$

Clearly $\mathcal{M}_\lambda \to \mathcal{M}$ when $\lambda \to 0$.

Stability Theorem. For any bounded open set $\tilde{\mathcal{O}}$ with $d_H(\mathcal{O}^c, \tilde{\mathcal{O}}^c) < \epsilon$, for any $x \in \mathcal{M}_\lambda(\tilde{\mathcal{O}})$, there is a $y \in \mathcal{M}_{\lambda'}(\mathcal{O})$ such that

$$d(x, y) \leq 2\sqrt{D\epsilon},$$

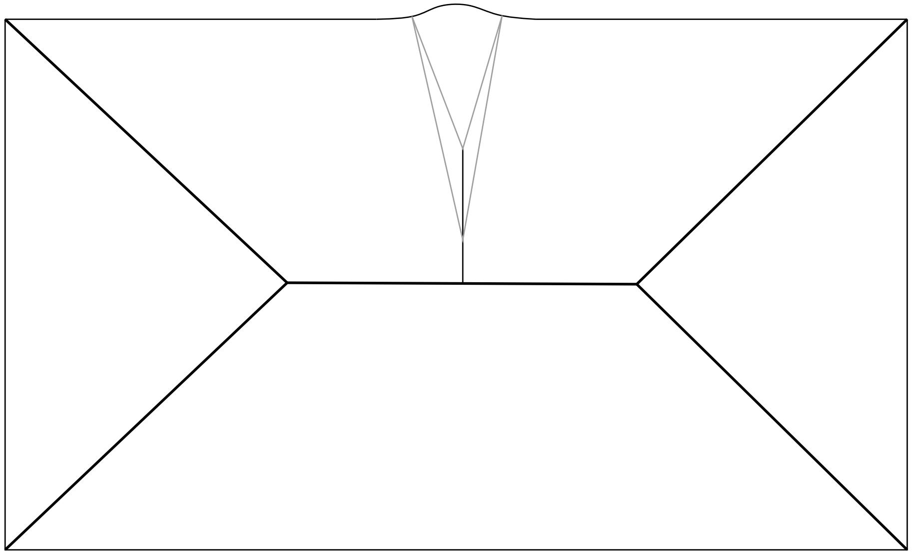
provided that

$$\lambda > 10\epsilon \qquad \text{and} \qquad \lambda'^2 < \lambda^2 - 150\sqrt{\epsilon}D^{3/2}$$

$\mathbf{F}_\beta(\mathcal{O})$, angle trimming

We define for any $\beta \geq 0$

$$\mathbf{F}_\beta(\mathcal{O}) = \left\{x \in \mathcal{O} : \|\nabla(x)\| \leq \beta\right\} = \left\{x \in \mathcal{O} : \cos(\theta(x)) \leq \beta\right\}.$$



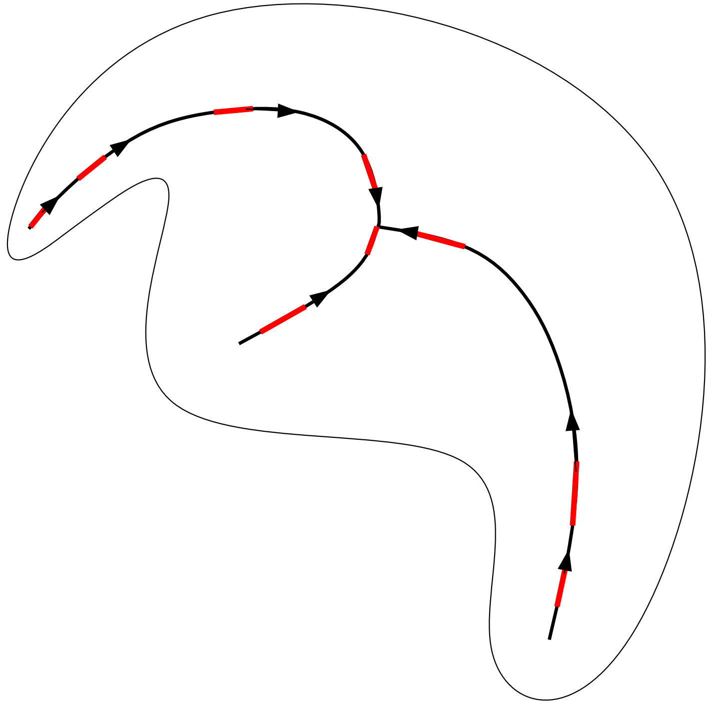
$\mathbf{F}$, the set of **critical points** of ∇ is

$$\mathbf{F} = \mathbf{F}_0$$

$\mathbf{G}_\beta(\mathcal{O})$ and weak feature size

$\mathbf{G}_\beta(\mathcal{O})$ is the closure of $\mathbf{F}_\beta$ under the flow $\mathcal{C}$.

$$\begin{aligned} \mathbf{G}_\beta(\mathcal{O}) &= \mathcal{C}(\mathbb{R}^+, \mathbf{F}_\beta(\mathcal{O})) \\ &= \left\{x \in \mathcal{O} : \exists t \in \mathbb{R}^+, \exists y \in \mathbf{F}_\beta(\mathcal{O}), x = \mathcal{C}(t, y)\right\} \end{aligned}$$



The **weak feature size** of $\mathcal{O}$, $\omega(\mathcal{O})$ is

$$\omega(\mathcal{O}) = d(\mathcal{O}^c, \mathbf{F}).$$

Every $x \in \mathcal{O}$ reaches $\mathbf{G}_\beta$ in finite time

Lemma. Let D be the diameter of $\mathcal{O}$. Then

$$\forall x \in O : \qquad \mathcal{C}(D/\beta^2, x) \in \mathbf{G}_\beta(\mathcal{O}).$$

Proof. If not,

$$\forall t \in [0, D/\beta^2] : \qquad \mathcal{C}(t, x) \notin \mathbf{F}_\beta(\mathcal{O})$$

which is the same as

$$\forall t \in [0, D/\beta^2] : \qquad \|\nabla(\mathcal{C}(t, x))\| > \beta.$$

Thus

$$\begin{aligned} \mathcal{R}(\mathcal{C}(D/\beta^2, x)) &= \mathcal{R}(x) + \int_0^{\frac{D}{\beta^2}} \nabla(\mathcal{C}(\tau, x))^2 d\tau \\ &> \mathcal{R}(x) + \int_0^{\frac{D}{\beta^2}} \beta^2 d\tau = \mathcal{R}(x) + D \end{aligned}$$

$\mathbf{G}_\beta$ carries the homotopy type of $\mathcal{O}$

Lemma. For any $\beta > 0$, $\mathbf{G}_\beta \simeq \mathcal{O}$.

Proof. Take the map

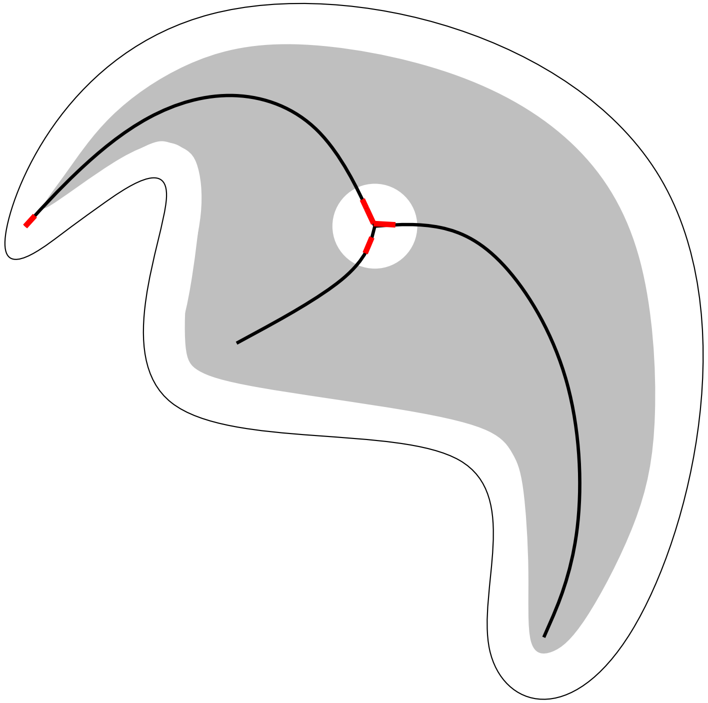
$$H : [0,1] \times \mathcal{O} \to \mathbf{G}_\beta(\mathcal{O})$$

given by

$$H(t,x) = \mathcal{C}(tD/\beta^2, x).$$

$\mathbf{F}_\beta$ is spread close to $\partial\mathcal{O}$ and $\mathbf{F}$

Lemma. $\forall \epsilon > 0,\ \exists \beta > 0, \mathbf{F}_\beta(\mathcal{O}) \subset (\mathcal{O}^c \cup \mathbf{F}) \oplus \mathbb{B}(0,\epsilon).$



Proof. By definition

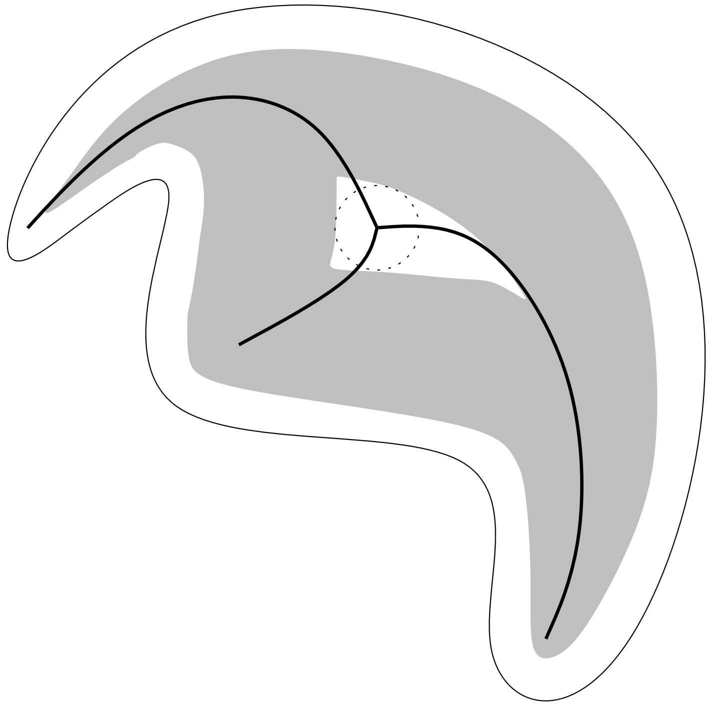
$$\mathbf{F} = \bigcap_{n\in\mathbb{N}, n\geq 1} \mathbf{F}_{1/n}(\mathcal{O}).$$

For any integer $n \geq 1$ let

$$K_n = \mathbf{F}_{1/n}(\mathcal{O}) \setminus ((\mathcal{O}^c \cup \mathbf{F}) \oplus \mathbb{B}(0,\epsilon))$$

Since $\mathbf{F}_{1/n}$ is closed for relative topology in $\mathcal{O}$, K_n is compact.
Also $K_n \subseteq K_{n-1}$ and $\bigcap_{n\geq 1} K_n = \emptyset$. Thus for some $N \in \mathbb{N}$, $K_N = \emptyset$.

Parameterizing the Flow



Let

$$\mathbf{D}_\epsilon = \left\{ x \in \mathcal{O} : \epsilon \leq \mathcal{R}(x) \leq \omega(\mathcal{O}) - \epsilon \right\}$$

Last Lemma shows that $\exists \beta > 0$, such that

$$\forall x \in \mathbf{D}_\epsilon, \qquad \beta < \|\nabla(x)\|.$$

Claim. For $r \in [\epsilon, \omega - \epsilon]$ and $x \in \mathbf{D}_\epsilon$, there is a unique $t \in [0, \omega/\beta^2]$ such that

$$\mathcal{R}(\mathcal{C}(t,x)) = \mathcal{R}(x) + r$$

Proof.

- $\mathcal{R}(\mathcal{C}(\omega/\beta^2)) \geq \omega - \epsilon$ (again by bounding the integral).
- $t \mapsto \mathcal{R}(\mathcal{C}(t,x))$ is increasing.

The map $\mathcal{C}_{\mathcal{R}}$

For $x \in \mathbf{D}_\epsilon$ and $r \in [0, \omega - \epsilon - \mathcal{R}(x)]$,

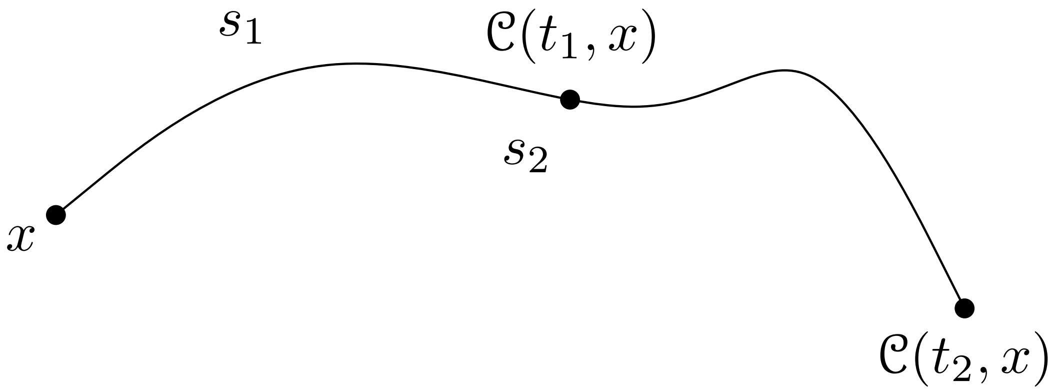
$$\mathcal{C}_{\mathcal{R}}(r, x) = \mathcal{C}(x, t),$$

for the unique t satisfying $\mathcal{C}(x, t) = r$.

Three Inequalities

Suppose $x, \mathcal{C}(t_1, x), \mathcal{C}(t_2, x) \in \mathbf{D}_\epsilon$. Let

$$r_1 = \mathcal{R}(C(t_1, x)) - \mathcal{R}(x) \qquad \text{and} \qquad r_2 = \mathcal{R}(\mathcal{C}(t_2, x)) - \mathcal{R}(x)$$



$$\begin{aligned}
\beta(t_2 - t_1) < &\quad s_2 - s_1 \quad \leq t_2 - t_1 \\
\beta(s_2 - s_1) < &\quad r_2 - r_1 \quad \leq s_2 - s_1 \\
\beta^2(t_2 - t_1) < &\quad r_2 - r_1 \quad \leq t_2 - t_1
\end{aligned}$$

Local Lipschitzness of $\mathcal{C}_{\mathcal{R}}$

Using these inequalities plus the general Lipschitz bound on $\mathcal{C}$, one can show.

Lemma. For any $\epsilon > 0$, using the appropriate $\beta > 0$

$$\forall x, x' \in \mathbf{D}_\epsilon, \quad \forall r \in [0, \omega - \epsilon - \mathcal{R}(x)], \quad \forall r' \in [0, \omega - \epsilon - \mathcal{R}(x')]$$

we have

$$\|\mathcal{C}_{\mathcal{R}}(r, x) - \mathcal{C}_{\mathcal{R}}(r', x')\| \leq \frac{1}{\beta}|r - r'| + \left(\left(1 + \frac{1}{\beta}\right) e^{\omega/\epsilon\beta^2} + \frac{1}{\beta}\right) \|x - x'\|$$

Extending $\mathcal{C}_{\mathcal{R}}$ beyond $\mathbf{D}_\epsilon$

If $\epsilon_1 < \epsilon_2$, then

$$\mathbf{D}_{\epsilon_2} \subset \mathbf{D}_{\epsilon_1}.$$

Then $\mathcal{C}_{\mathcal{R}}$ defined on $\mathbf{D}_{\epsilon_1}$ extends the map $\mathcal{C}_{\mathcal{R}}$ defined on $\mathbf{D}_{\epsilon_2}$.

So, $\mathcal{C}_{\mathcal{R}}$ is defined unambiguously on

$$\bigcup_{\epsilon>0} \mathbf{D}_\epsilon = \mathcal{O} \setminus \overline{\mathcal{O}_\omega}.$$

$\mathcal{C}_{\mathcal{R}}$ is locally Lipschitz and therefore continuous.

Homotopy Type of $\mathcal{O}_\rho$

Theorem. If $\rho < \omega$, then $\overline{\mathcal{O}_\rho}$ is a deformation retract of $\mathcal{O}$. Furthermore $\mathcal{O} \simeq \mathcal{O}_\rho$.

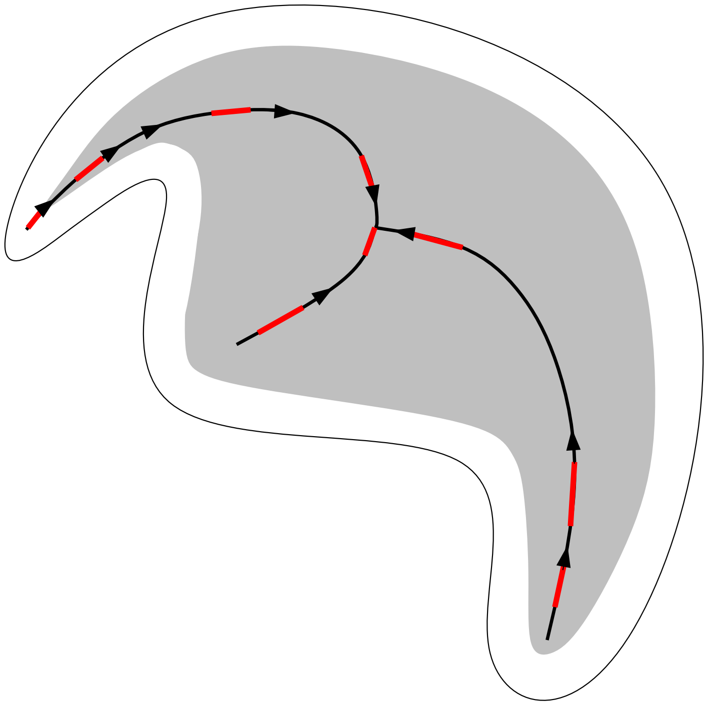
Proof. Use the map $H_\rho : [0,1] \times \mathcal{O} \to \overline{\mathcal{O}_\rho}$:

$$(t,x) \mapsto \begin{cases} \mathcal{C}_{\mathcal{R}}(t(\rho - \mathcal{R}(x)), x) & \text{if } \mathcal{R}(x) \leq \rho \\ x & \text{if } \mathcal{R}(x) \geq \rho \end{cases}$$

This is a deformation retract of $\mathcal{O}$ to $\overline{\mathcal{O}_\rho}$.

If we use $H_{(\rho+\omega)/2}$ we get the homotopy equivalence between $\mathcal{O}$ and $\mathcal{O}_\rho$.

$\mathbf{G}_\beta^\rho$: Trimming $\mathbf{F}_\beta$ with $\mathcal{O}_\rho$ Before Closure



Let $\mathbf{G}_\beta^\rho$ be the closure under $\mathcal{C}$ of $\mathcal{O}_\rho \cap \mathbf{F}_\beta(\mathcal{O})$:

$$\mathbf{G}_\beta^\rho = \mathcal{C}(\mathbb{R}^+, \mathcal{O}_\rho \cap \mathbf{F}_\beta(\mathcal{O}))$$

Lemma. If $\rho < \omega$ and $\beta > 0$, $\mathbf{G}_\beta^\rho \simeq \mathcal{O}$.

Proof. $\mathbf{G}_\beta^\rho \subset \overline{\mathcal{O}_\rho} \simeq \mathcal{O}$ (starts $> \rho$ away from boundary and goes farther).

So, only need to show $\mathbf{G}_\beta^\rho \simeq \overline{\mathcal{O}_\rho}$:

$$(t, x) \mapsto \mathcal{C}(tD/\beta^2, x)$$

Homotopy Type of $\mathcal{M}_\lambda$

Lemma. If $\lambda < \omega$, then $\exists \beta > 0$, such that

$$\mathbf{G}_\beta^\lambda \subset \mathcal{M}_\lambda$$

Proof. Let $\epsilon = \min\left(\frac{\lambda}{2}, \frac{\omega-\lambda}{2}\right)$. There exists $\beta_0 > 0$ such that:

$$\forall 0 < \beta \leq \beta_0, \quad d(\mathcal{O}^c, \mathcal{O}_\lambda \cap \mathbf{F}_\beta(\mathcal{O})) > \lambda + \epsilon$$

Choose β small enough so that

$$(\lambda + \epsilon)\sqrt{1 - \beta^2} > \lambda$$

$$x \in \mathcal{O}_\lambda \cap \mathbf{F}_\beta \quad \Rightarrow \quad \begin{array}{l} \mathcal{R}(y) > \lambda + \epsilon \\ 1 - \frac{\mathcal{F}(x)^2}{\mathcal{R}(x)^2} \leq \beta^2 \end{array} \quad \Rightarrow \quad \mathcal{F}(y) > \lambda$$

So, $\mathcal{O}_\lambda \cap \mathbf{F}_\beta(\mathcal{O}) \subset \mathcal{M}_\lambda$. The rest follows from fact that $t \mapsto \mathcal{F}(t, x)$ is increasing.

Main Theorem

Theorem. If $\lambda < \omega(\mathcal{O})$, then $\mathcal{M}_\lambda(\mathcal{O}) \simeq \mathcal{O}$.

Proof.

- Since $t \mapsto \mathcal{F}(t,x)$ is increasing, flow doesn't leave $\mathcal{M}_\lambda$.
- Now take the value β from previous lemma for which

$$\mathbf{G}_\beta^\lambda \subset \mathcal{M}_\lambda$$

- By definition, flow doesn't leave $\mathbf{G}_\beta^\lambda$.
- We have seen that $\mathcal{O}$ flows to $\mathbf{G}_\beta$ under

$$(t,x) \mapsto \mathcal{C}(tD/\beta^2, x)$$

It only remains to observe that $\mathcal{M}_\lambda \subset \mathcal{O}_\lambda$ since $\mathcal{F}(x) \leq R(x)$.