Undirected st-Connectivity in Log-Space by Omer Reingold

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USTCON and STCON

st-connectivity (STCON)
input: *directed* graph G, and vertices s and t
question: is there is a path *from s to t* in G?

Complete for NL (nondeterministic log-space)

undirected st-connectivity (USTCON)
input: undirected graph G, and vertices s and t
question: is there a path between s and t in G?

complete for SL (symmetric nondeterministic log-space [Lewis and Papadimitriou '82])

"symmetric" means: if (q_1, σ, q_2) is a transition, then (q_2, σ, q_1) is also a transition.

What is known . . .

$$\mathsf{L} \subseteq_{(1)} \mathsf{SL} \subseteq_{(2)} \mathsf{RL} \subseteq_{(3)} \mathsf{NL} \subseteq_{(4)} \mathsf{L}^2$$

- (1) and (3): Trivial!
- (4): Savitch's theorem [Savitch '70]
- (2): [Aleliunas, Karp, Lipton, Lovász '79]

Idea for (2): A random walk started anywhere in graph reaches every reachable vertex in poly(n) *expected* number of steps!

Also known:

- USTCON $\in \mathsf{L}^{3/2}$ [Nisan, Szémeredi, Wigderson '89]
- RL $\subseteq \mathsf{L}^{3/2}$ [Saks, Zhou '99]
- USTCON $\in \mathsf{L}^{4/3}$ [Armoni, Ta-Shma, Wigderson, Zhou '??]
- USTCON $\in$ L [Reingold '05] (this talk)

Expanders

Given a graph $G = (V, E)$ (can have loops and multiple edges) and $S \subseteq V$, the **edge boundary** of S is

$$\partial S = E(S, \overline{S}) = \{uv : u \in S, v \in \overline{S}, uv \in E\}.$$

The **expansion parameter** of a graph G is defined as

$$h(G) = \min_{S \subseteq V, |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}.$$

A d-regular graph is an ϵ**-expander** if $h(G) \geq \epsilon$.

Observation: Every connected graph G has $h(G) \geq 1/n$.

Eigenvalue expansion and st-connectivity

Let G be a d-regular graph with adjacency matrix A. The **normalized adjacency matrix** $\hat{A}$ of G is $\frac{1}{d}A$. (Transition probability matrix for random walk on G)

The vector $\mathbf{x} = (\underbrace{1, 1, \ldots, 1}_{d})^T$ is an eigenvector for A with eigenvalue 1:

$$\hat{A}\mathbf{x} = \mathbf{x}.$$

Since $\hat{A}$ is symmetric, all eigenvalues of $\hat{A}$ are real. Turns out that for every eigenvalue λ of $\hat{A}$, $|\lambda| \leq 1$.

Define

$$\lambda(G) := \textit{second largest eigenvalue} \text{ of } G.$$

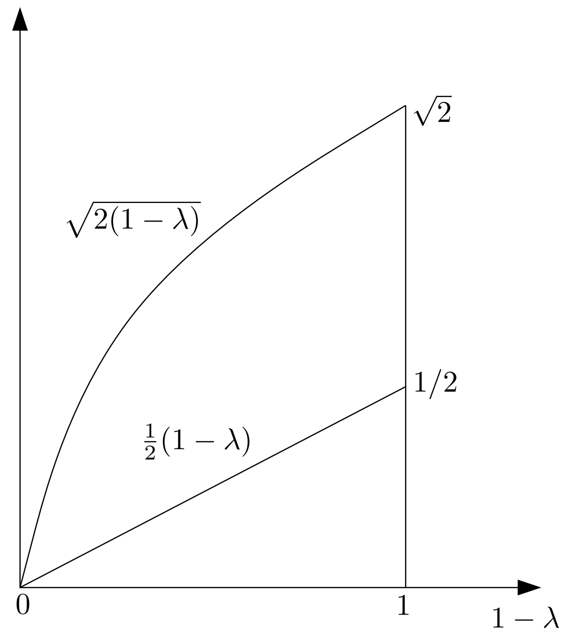
$1 - \lambda(G)$, the **spectral gap**, closely represents $h(G)$.

More on expanders...

Theorem. [Tanner '84, Alon and Milman '86]

$$d \cdot \frac{1-\lambda}{2} \leq h(G) \leq d\sqrt{2(1-\lambda)},$$

where $\lambda = \lambda(G)$.



Notation: An (n, d, λ)-graph G is an n-vertex, d-regular graph with $\lambda(G) = \lambda$.

Constant expansion $\Rightarrow$ logarithmic diameter

Corollary. $\forall \lambda < 1$, $\exists \epsilon = \epsilon(d, \lambda) > 0$, such that for every subset S of size $\leq n/2$ of vertices of an (n, d, λ)-graph,

$$|S| + |\partial S| \geq (1 + \epsilon)|S|.$$

Proposition. A connected (n, d, λ)-graph has diameter $O(\log n)$.

Proof. For some $\ell = O(\log n)$ any vertex s has $> n/2$ vertices within distance ℓ.

Rotation map representation of Graphs

Let G be a d-regular graph. Assume that the edges incident to each vertex are labeled $1, \ldots, d$ in an arbitrary but fixed way.

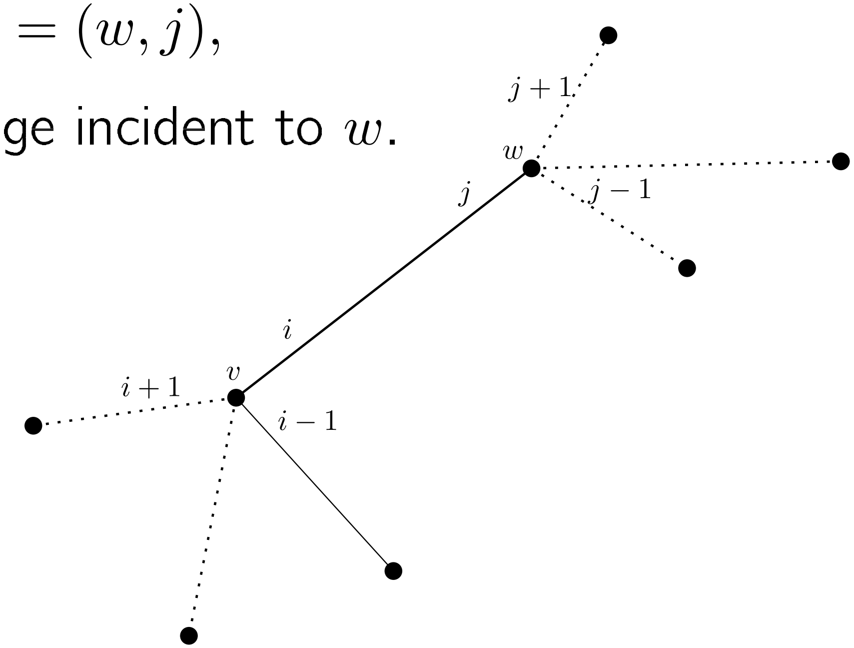
For a d-regular n vertex graph G, the **rotation map**

$$\mathrm{Rot}_G : [n] \times [d] \longrightarrow [n] \times [d],$$

is defined by

$$\mathrm{Rot}_G(v, i) = (w, j),$$

if ith edge incident to v is the jth edge incident to w.



The BIG picture

Any transformation on input graph that maintains the connected components of G (**safe transformation**), does not affect the answer to an instance of st-connectivity.

Outline of the algorithm:

1. Turn input graph G (safely) into a d-regular graph (constant d).
2. Improve expansion of G (safely and keeping the degree bounded) till every component of G becomes an expander.
3. Try all possible $O(\log n)$ possible rotation-map paths (polynomially many) starting at s and check if t is reached.

Improving expansion: Powering

Given d-regular graph G with adjacency matrix A, the tth power of G, G^t, is the d^t-regular graph with adjacency matrix A^t.

Simple exercise: $A^t = [a_{ij}]$, where a_{ij} is the number of distinct paths of length $\leq t$ in G from i to j.

The rotation map of G^t:

$$\text{Rot}_{G^t}(v_0, (a_1, a_2, \ldots, a_t)) = (v_t, (b_t, b_{t-1}, \ldots, b_1)),$$

where $(v_i, b_i) = \text{Rot}_G(v_{i-1}, a_i)$.

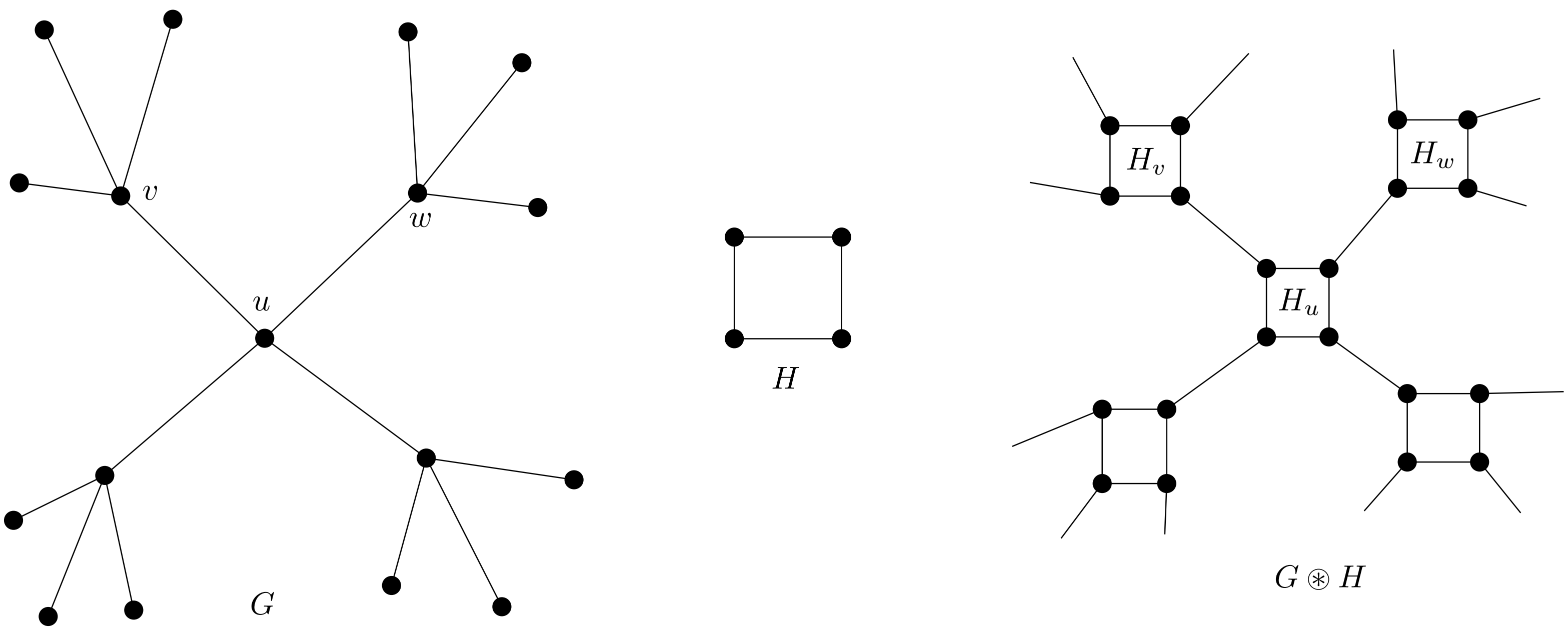
Proposition. If G is an (n, d, λ)-graph, then G^t is an (n, d^t, λ^t)-graph.

Problem. Vertex degrees of G^t can be to high for a good enough t.

Reducing degrees: replacement product

If G is a D-regular graph on $[N]$ and H is a d-regular graph on $[D]$.

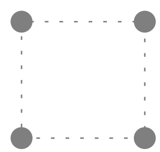
The **replacement product** of G and H, $G \circledast H$, replaces every vertex v of G by a copy H_v of H, every vertex of H_v is further made incident to one edge incident to v in G.



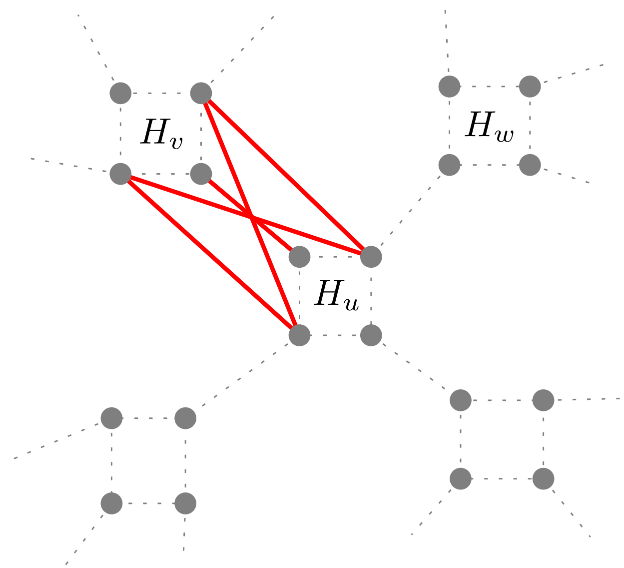
Observation: $G \circledast H$ is $(d+1)$-regular.

Pushing further: zig-zag product

Given a D-regular graph G on $[N]$ and a d-regular graph H on $[D]$, the **zig-zag** product $G \ltimes H$ is graph with the same vertex set as $G \circledast H$, in which u and v are adjacent if u and v are connected by a path of length 3 in $G \circledast H$ as follows.



H



$G \circledast H$

Observation: $G \ltimes H$ is d^2-regular.

Rotation map of $G \ltimes H$

Having Rot_G and Rot_H at hand $\text{Rot}_{G \ltimes H}$ is log-space computable.

$\text{Rot}_{G \ltimes H}((v, a), (i, j))$:

1. Let $(a', i') = \text{Rot}_H(a, i)$.
2. Let $(w, b') = \text{Rot}_G(v, a')$.
3. Let $(b, j') = \text{Rot}_H(b', j)$.
4. Output $((w, b), (j', i'))$.

Zig-zag product doesn't loose much expansion

Theorem. [Reingold, Vadhan, Wigderson '01] If G is an (N, D, λ)-graph and H is a (D, d, α) graph, then

$$1 - \lambda(G \ltimes z) \geq \frac{1 - \alpha^2}{2} \cdot (1 - \lambda).$$

I.E. The zig-zag product drops the spectral gap by at most a factor that only depends on $\lambda(H)$.

Main Transformation

Transformation $\mathcal{T}$:
input:

- G: a D^{16}-regular graph on $[N]$
- H: a D-regular graph on $[D^{16}]$

output: G_ℓ, where ℓ is the smallest integer satisfying

$$\left(1 - \frac{1}{DN^2}\right)^{2^\ell} < 1/2,$$

$G_0 = G$, and for $i > 0$,

$$G_i = (G_{i-1} \ltimes H)^8.$$

I.E.

$$\mathcal{T}(G, H) = ((((G \ltimes H)^8 \ltimes H)^8) \ldots \ltimes H)^8.$$

Properties of $\mathcal{T}$

- $G_1, \ldots, G_\ell$ are D^{16}-Regular graphs over $[N] \times ([D^{16}])^i$, resp.
- If D is constant, $\ell = O(\log N)$, and G_ℓ has poly(N) vertices.

Lemma If $\lambda(H) \leq 1/2$ and G is connected and non-bipartite, then $\lambda(G_\ell) \leq 1/2$.

Proof. Use the bounds on $\lambda(G^t)$ and $\lambda(G \ltimes H)$ plus the following result from Alon and Sudakov.

Theorem. [Alon, Sudakov '00] If G is a D-regular, connected, non-bipartite graph on $[N]$, then

$$\lambda(G) \leq 1 - \frac{1}{DN^2}.$$

$\mathcal{T}$ operates on components of G independently

We want to solve st-connectivity and not connectivity, so we need $\mathcal{T}$ to work on disconnected graphs by transforming each of their components.

Lemma. If $S \subseteq [N]$ is a connected component of G then

$$\mathcal{T}(G|_S, H) = T(G, H)|_{S \times ([D^{16}])^{\ell}}.$$

Proof. Follows from definitions.

$\mathcal{T}$ is a log-space transformation

Each chunk of output (to be overwritten by the following chunks) is of the form

$$[(\overline{v}, \overline{a}), \mathsf{Rot}_{G_\ell}(\overline{v}, \overline{a})],$$

where

$$\overline{v} \in [N] \times ([D^{16}])^\ell \qquad \text{and} \qquad \overline{a} \in [D^{16}].$$

Thus chunks have size $O(\log N)$.

By definition of $\mathcal{T}$, the algorithm needs to $\ell = O(\log n)$ times follow the rotation maps recursively, but each time only for $O(1)$ steps. This is all the space needed to compute Rot_{G_ℓ}.

We still need a good H

There are several explicit constructions available for H . . .

Proposition. [Alon and Roichman '94, or many others . . .] There exists some constant D_0 and a $((D_0)^{16}, D_0, 1/2)$-graph!

Putting everything together

The algorithm relies on a $(D_0^{16}, D_0, 1/2)$-graph H_0, where D_0 is a constant.

1. On input G, s, t, construct a D_0^{16}-regular graph G' such that all its connected components are non-bipartite (easy and standard techniques). Let s' and t' be the vertices corresponding to s and t in G'.
2. Let $G'' = \mathcal{T}(G', H_0)$ and let s'', t'' be representatives of s and t in G''.
3. Enumerate the rotation map of every path of length $O(\log N)$ from s and see if it reaches t.

Each of the enumerated paths takes space

$$O(\log D_0^{16} \cdot \log N) = O(\log N).$$