1. Logic

Prove (uniqueness of complement law) for a Boolean Algebra.

A Boolean Algebra is a set $B$ containing special elements 1 and 0 together with binary operators $+$ and $\cdot$ and a unary operator $'$ which satisfy the following axioms for all $x, y, z \in B$.

- (commutative laws) $\quad x \cdot y = y \cdot x$
  $\quad x + y = y + x$
- (associative laws) $\quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$
  $\quad x + (y + z) = (x + y) + z$
- (distributive laws) $\quad x \cdot (y + z) = x \cdot y + x \cdot z$
  $\quad x + (y \cdot z) = (x + y) \cdot (x + z)$
- (identity laws) $\quad x \cdot 1 = x$
  $\quad x + 0 = x$
- (complement laws) $\quad x + x' = 1$
  $\quad x \cdot x' = 0$

The special elements 0 and 1 is usually called called zero and unity, respectively. $x'$ is the complement of $x$. Often $x \cdot y$ is written as $x \cdot y$. Given these axioms we can derive (prove) the following “rules” of Boolean arithmetic.

**Theorem BA:** For elements 0, 1, $x$ and $y$ of a Boolean algebra:

- (complement of identities) $\quad 0' = 1$
  $\quad 1' = 0$
- (involution law) $\quad (x')' = x$
- (uniqueness of complement law) $\quad$ if $\quad x + x' = 1$ and $x \cdot x' = 0$
  then $x' = x^*$
idempotent laws) \[ x + x = x \]
\[ x \cdot x = x \]

(null laws) \[ x + 1 = 1 \]
\[ x \cdot 0 = 0 \]

(absorption laws) \[ x + (x \cdot y) = x \]
\[ x \cdot (x + y) = x \]

(De Morgan’s Laws) \[ (x + y)' = x' \cdot y' \]
\[ (x \cdot y)' = x' + y' \]

Why is this theorem different from ThmB? ThmB started with a definition for +, \cdot and \,' and then stated all the properties that are part of the definition of a Boolean algebra and ThmBA. Here we start with the definition of an algebraic system and state that the other properties follow. In fact, we will see soon that, if \( B = \{0, 1\} \), then the defining rules for +, \cdot and \,' must be those of section 2.

2. Linear Algebra

[2.1] Prove ThmR following the hints.

**Theorem R.** For any \( m \times n \) matrix \( A \), there exist \( m \times m \) and \( n \times n \) permutation matrices \( P \) and \( Q \), an \( m \times m \) matrix \( L \) and an \( m \times n \) matrix \( U \), and an integer \( 0 \leq r \leq \min(m, n) \) such that

\[
\text{(FCT)} \quad PAQ = LU.
\]

For \( p = m-r \) and \( q = n-r \), \( L \) is \( m \times m \) unit lower triangular and of the form

\[
\text{(ULT)} \quad L = \begin{bmatrix}
L_r & 0_p \\
0_pr & I_p
\end{bmatrix},
\]

where \( L_r \) \( r \times r \); and \( U \) is \( m \times n \) taking the form

\[
\text{(U)} \quad U = \begin{bmatrix}
U_r & U_q \\
0_pr & 0_{pq}
\end{bmatrix}
\]

where \( U_r \) is \( r \times r \) upper triangular with nonzero diagonal entries and \( U_{rq} \) is an \( r \times q \) matrix. Possibly \( r, p, \) or \( q \) can be zero; in this case any submatrix with one of these indices is empty (does not exist).
First some comments, then some hints on the proof. First note that if \( A = 0 \) (the trivial case), \( A = I_m0_{m \times n} \), and the theorem holds. The integer \( r \) is called the rank of the matrix \( A \). It is not hard to prove from this theorem that \( r \) is both the maximum number of independent rows and columns of \( A \). Furthermore if \( A \) has \( r = \min(m,n) \), \( r \) is said to have full rank. In this case, either \( p = 0 \) or \( q = 0 \).

**Proof.** Use induction on \( k = \min(m,n) \). When \( k = 1 \), the matrix \( A \) is a single row or column. The \( A = 0 \) case is handled as above. If \( A 0 \neq 0 \), pick \( P \) and \( Q \) to move a nonzero into the \((1,1)\) position and check that the factorization (FCT) holds. Assuming the case \( k = j \), extend the factorization to the case \( k = j +1 \) similar to the proof of Theorem LU.

**Endproof.**

[2.2]. Apply ThmR to analyze the solution to the system \( Ax = b \) when \( A \) is \( m \times n \) and \( b \) is a column \( n \)-vector. Is there always a solution? Is there a solution for some \( b \)? What about \( b = 0 \) (homogeneous case)?

3. **Trees**

Do either exercise2 or exercise3 in the tree chapter3 (it is logically correct to do both (correctly)).

4. **Graphs**

Prove Thm5 below using the hints.

**Theorem 5.** A graph \( G = (X,E) \) is a k-tree if and only if

(i) \( G \) is connected;
(ii) \( G \) has a k-clique but no \( k + 2 \) clique;
(iii) every minimal \( x,y \) separator is a k-clique.

**Proof.**

**Here are some hints.**

Use induction on \( n = |X| \). First show the conditions are necessary. The cases \(|X| = k \) and \(|X| = k + 1 \) are fairly straightforward. Assume the case \( n \) and show the case \( n + 1 \) by picking off a vertex \( x \) such that \( \text{adj}(x) \) is a k-clique and considering \( G(X-x) \). You can use Prop3 to help show that (iii) holds in \( G \); be careful to consider both cases that \( x \) is in \( S \) or not in \( S \), the candidate separator in \( G \).

Now show the conditions are sufficient. To show that \( G \) is a k-tree, use induction on \( n \); determine the base case, and assume the cases \( m \leq n \). For the case \( n + 1 \), show that there is some vertex \( x \) such that \( \text{adj}(x) \) is a k-clique and that \( G(X-x) \) is a k-tree. Let \( S \) be some minimal \( x,y \) separator. Show, using Prop3, that the conditions hold in the leaf \( L_x \); hence
L_{x} is a k-tree. Now use Prop2 to get the existence of two separate vertices which adjacency sets are k-cliques. Not both can be in S; let z be the one not in S. Show that \text{adj}(z) is a k-clique in G and that G(X-z) is a k-tree again because the conditions hold in G(X-z).

Endproof.

5. Graphs and Matrices

Complete [5.1] –[5.6].

Let D = (X, E) with n, n', m and m' defined as in the graph case. The major difference here is that we will be over the field \( \mathbb{R} \), rather than \( \mathbb{F}_2 \) so \( 1 \neq -1 \). Our matrices will have as entries numbers from the set \{ -1, 0, 1 \}. Another difference is that xy and yx can both be (directed) edges in E. An oriented cycle (or-cycle), c, is a cycle in the undirected graph G(D) or a cycle of the form [x, y, x], and an oriented cut-set (or-cut-set), k, is a cut-set of G(D) and includes both xy and yx if they are in E and xy is in the cut-set of G. We will drop the adjective “oriented” if the context is clear. The orientation is arbitrary; a direction is specified by taking a directed edge from c or k, and the remaining edges are consistent or opposed to the orientation. When making a cycle or cut-set vector, v, v has one orientation and \(-v\) has the opposite orientation.

A spanning tree, T, of D is a spanning tree of G(D), but, if xy is a branch and yx is in F, then yx is a chord. As before branches give rise to f-cuts and chords give rise to f-cycles; the orientation of these f-cuts and f-cycles is taken from the orientation of the branches and chords.

[5.1] Define the matrices \( K_T \) and \( C_T \) in analogy with the graph case. Use \(-1\) when the f-cycle edges or f-cut edges are opposed to the branch and chord orientation. Show that \( K_T = [I_n, K] \) and \( C_T = [C, I_m] \); the \(-1\) entries are only in K and C. Derive \( K_T \) and \( C_T \) for the digraph of Fig1c (find a spanning tree first). Show that the rows of both \( K_T \) and \( C_T \) are independent.

[5.2] Prove that \( Z = C_T(K_T)^T = 0 \) still holds. State (and prove) a Theorem analogous to Thm1.

[5.3] State (and prove) a Theorem analogous to Thm2.

[5.4] State (and prove) a Theorem analogous to Thm3 and the analogue of Cor1.

[5.5] Prove that the rows in \( M = \begin{bmatrix} I & K \\ C & I \end{bmatrix} \) are independent (hence the situation in Example2 does not happen). Start by making the graph of Fig2 into a digraph and work through the details on this example. Hint: if \( Mx = 0 \) for some nonzero \( x = [a; b] \) partitioned as in M, then \( x^T Mx = 0 \). Is this possible for nonzero \( x \)?
[5.6] Define $A_0$ and $A$ in the digraph case and state and prove a Theorem analogous to Thm4. Make Example 3 into a digraph and find $A_0$ and $A$ analogous to the graph case.

6. Extra Credit: Discrete Probability

Prove the second inequality of (LLN5) below. Some hints are given in chapt6.

We will now consider in some detail how close the random variable, $S_n$, is to $np$ for large $n$ (large numbers). To be precise, we will consider the probability of the event $E(n, \varepsilon) = \{| S_n/n - p | < \varepsilon \}$ and show that for any (small) $\varepsilon$

(LLN4) $p(E(n, \varepsilon)) \to 1$ as $n \to \infty$.

To do this we must consider the binomial distribution in some detail.

We will prove (LLN4) by showing that

(LLN5) $p\{S_n \geq n(p + \varepsilon)\} \to 0$ and $p\{S_n \leq n(p - \varepsilon)\} \to 0$ as $n \to \infty$.