Example

\[ L = \{a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]

Example

\[ L_1 = \{x \mid x \text{ is a positive even integer} \} \]

L is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[ L_1 \cup L_2 \]
\[ L_1 \cap L_2 \]
\[ L_1 L_2 \]
\[ L_1^* \]
\[ L_1^* \]

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1r_2$ is r.e. denoting $L_1L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure

complementation:
$L_1$ is reg. lang.
$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$
Construct $M'$ s.t.
final states in $M$ are
nonfinal states in $M'$
final states in $M$ are
nonfinal states in $M'$
$\Rightarrow$ closed under complementation

intersection:
$L_1$ and $L_2$ are reg. lang.
$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.
$L_1 = L(M_1)$ and $L_2 = L(M_2)$
$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
$Q' = (Q \times P)$
$\delta'$:
$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$
$\Rightarrow$ closed under intersection
Example:

Regular languages are closed under

- **reversal** \( L^R \)
- **difference** \( L_1 - L_2 \)
- **right quotient** \( L_1 / L_2 \)
- **homomorphism** \( h(L) \)

**Right quotient**

**Def:** \( L_1 / L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \} \)

**Example:**

\[
L_1 = \{ a^*b^* \cup b^*a^* \} \\
L_2 = \{ b^n | n \text{ is even}, n > 0 \} \\
L_1 / L_2 =
\]

**Theorem** If \( L_1 \) and \( L_2 \) are regular, then \( L_1 / L_2 \) is regular.

**Proof** (sketch)

\[ \exists \text{ DFA } M = (Q, \Sigma, \delta, q_0, F) \text{ s.t. } L_1 = L(M). \]

Construct DFA \( M' = (Q, \Sigma, \delta', q_0', F') \)

For each state \( i \) do

- Make \( i \) the start state (representing \( L_i' \))
- if \( L_i' \cap L_2 \neq \emptyset \) then
  - put \( q_i \) in \( F' \) in \( M' \)

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) =$

$h(ab^*) =$

Questions about regular languages:

$L$ is a regular language.

- Given $L, \Sigma, w \in \Sigma^*$, is $w \in L$?

- Is $L$ empty?

- Is $L$ infinite?

- Does $L_1 = L_2$?
Identifying Nonregular Languages

If a language L is finite, is L regular?

If L is infinite, is L regular?

- \( L_1 = \{a^n b^m | n > 0, m > 0\} = a^* b^* \)
- \( L_2 = \{a^n b^n | n > 0\} \)

Prove that \( L_2 = \{a^n b^n | n > 0\} \) is ?

- Proof:
**Pumping Lemma:** Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
x y^i z &\in L \text{ for all } i \geq 0
\end{align*}$$

**Meaning:** Every long string in $L$ (the constant $m$ above corresponds to the finite number of states in $M$ in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in $L$.

**To Use the Pumping Lemma to prove $L$ is not regular:**

- **Proof by Contradiction.**
  - Assume $L$ is regular.
  - $\Rightarrow$ $L$ satisfies the pumping lemma.
  - Choose a long string $w$ in $L$, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  - Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^i z \in L \forall i \geq 0$.
  - The pumping lemma does not hold. Contradiction! $\Rightarrow$ $L$ is not regular. QED.

**Example** $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  - Assume $L$ is regular.
  - $\Rightarrow$ the pumping lemma holds.
  - Choose $w = $ where $m$ is the constant in the pumping lemma. (Note that $w$ must be chosen such that $|w| \geq m$.)
  - The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with $c b^m$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

It should be true that $xy^i z \in L$ for all $i \geq 0$. 
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  ⇒ the pumping lemma holds.
  Choose $w = \quad$
  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with the rest of the string $b^{m+s} c^s$.
  This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

Example $\Sigma = \{a, b\}, L = \{w \in \Sigma^* | n_a(w) > n_b(w)\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  ⇒ the pumping lemma holds.
  Choose $w = \quad$
  So the partition is:
Example $L = \{a^3b^nc^{n-3} \mid n > 3\}$

$L$ is not regular.

- **Proof:**

  Assume $L$ is regular. $\Rightarrow$ the pumping lemma holds.

  Choose $w = a^3b^mc^{m-3}$ where $m$ is the constant in the pumping lemma. There are three ways to partition $w$ into three parts, $w = xyz$. 1) $y$ contains only $a$'s 2) $y$ contains only $b$'s and 3) $y$ contains $a$'s and $b$'s

  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide $w$ into three parts s.t. the pumping lemma contraints were true).

  **Case 1:** ($y$ contains only $a$'s). Then $x$ contains $0$ to $2$ $a$'s, $y$ contains $1$ to $3$ $a$'s, and $z$ contains $0$ to $2$ $a$’s concatenated with the rest of the string $b^mc^{m-3}$, such that there are exactly $3$ $a$’s. So the partition is:

  $$x = a^k \quad y = a^j \quad z = a^{3-k-j}b^mc^{m-3}$$

  where $k \geq 0$, $j > 0$, and $k + j \leq 3$ for some constants $k$ and $j$.

  It should be true that $xy^iz \in L$ for all $i \geq 0$.

  $xy^2z = (x)(y)(y)(z) = (a^k)(a^j)(a^j)(a^{3-j-k}b^mc^{m-3}) = a^{3+j}b^mc^{m-3} \notin L$ since $j > 0$, there are too many $a$'s. Contradiction!

  **Case 2:** ($y$ contains only $b$'s) Then $x$ contains $3$ $a$’s followed by $0$ or more $b$’s, $y$ contains $1$ to $m-3$ $b$’s, and $z$ contains $3$ to $m-3$ $b$’s concatenated with the rest of the string $c^{m-3}$. So the partition is:

  $$x = a^3b^k \quad y = b^j \quad z = b^{m-k-3}c^{m-3}$$

  where $k \geq 0$, $j > 0$, and $k + j \leq m - 3$ for some constants $k$ and $j$.

  It should be true that $xy^iz \in L$ for all $i \geq 0$.

  $xy^2z = a^3b^{m-j}c^{m-3} \notin L$ since $j > 0$, there are too few $b$’s. Contradiction!

  **Case 3:** ($y$ contains $a$’s and $b$’s) Then $x$ contains $0$ to $2$ $a$’s, $y$ contains $1$ to $3$ $a$’s, and $1$ to $m-3$ $b$’s, $z$ contains $3$ to $m-1$ $b$’s concatenated with the rest of the string $c^{m-3}$. So the partition is:

  $$x = a^{3-k} \quad y = a^kb^j \quad z = b^{m-j}c^{m-3}$$

  where $3 \geq k > 0$, and $m - 3 \geq j > 0$ for some constants $k$ and $j$.

  It should be true that $xy^iz \in L$ for all $i \geq 0$.

  $xy^2z = a^3b^ja^kb^mc^{m-3} \notin L$ since $j, k > 0$, there are $b$’s before $a$’s. Contradiction!

  $\Rightarrow$ There is no partition of $w$.

  $\Rightarrow L$ is not regular!. QED.
To Use Closure Properties to prove L is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**
  
  Assume L is regular.
  
  Apply closure properties to L and other regular languages, constructing L' that you know is not regular.
  
  closure properties $\Rightarrow$ L' is regular.
  
  Contradiction!
  
  L is not regular. QED.

**Example** \( L = \{a^3b^n c^n | n > 3\} \)

L is not regular.

- **Proof:** (proof by contradiction)
  
  Assume L is regular.
  
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  
  \( h(a) = a \quad h(b) = a \quad h(c) = b \)
  
  \( h(L) = \)
**Example** \( L = \{ a^n b^m a^m | m \geq 0, n \geq 0 \} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  
  Assume \( L \) is regular.

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**Example:** \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.

- **Proof:**
  
  Assume \( L_1 \) is regular.

  Goal is to try to construct \( \{ a^n b^n | n > 0 \} \) which we know is not regular.

  Let \( L_2 = \{ a^* \} \). \( L_2 \) is regular.

  By closure under right quotient, \( L_3 = L_1 \setminus L_2 = \{ a^n b^n a^p | 0 \leq p \leq n, n > 0 \} \) is regular.

  By closure under intersection, \( L_4 = L_3 \cap \{ a^* b^* \} = \{ a^n b^n | n > 0 \} \) is regular.

  Contradiction, already proved \( L_4 \) is not regular!

  Thus, \( L_1 \) is not regular. QED.