In mathematics, a perfect square is an integer that is the square of an integer; in other words, it is the product of some integer with itself. For example, 9 is a perfect square, since it can be written as $(3 \times 3)$.

Suppose that $(a/b)$ and $(c/d)$ are arbitrary rational numbers, where we have $a, b, c, d \in \mathbb{Z}$ and $(a/b)^2 + (c/d)^2 = 1$.

Now, let $y = (bc/d)^2$.

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Harsh claims that $y \in \mathbb{Q}$. Is he correct? Why?

**Answer:** Yes, he is correct. Since $y$ could be expressed as fraction of integers, $(bc/d)^2$.

2, 3

Abhishek claims that $y \in \mathbb{Z}$. Is he correct? State your answer as a theorem. Prove it.

**Answer:** Yes, he is correct. Since

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 1$$

multiply by $b^2$ to get

$$b^2 - a^2 = \left(\frac{bc}{d}\right)^2$$
Therefore, \( y \) could be represented as \( b^2 - a^2 \). Because both \( a, b \in \mathbb{Z} \), and thus \( a^2, b^2 \in \mathbb{Z} \).

\[
\therefore y = b^2 - a^2 \in \mathbb{Z}.
\]

4, 5

Professor Boo Barkee claims that \( y \) is a perfect square. Are they correct? State your answer as a theorem. Prove it.

**Answer:** We derived from last proof that \( b^2 - a^2 \in \mathbb{Z} \). Now, we claim that \( b^2 - a^2 \) must be a perfect square.

(intuitively.. "You cannot square a non-integer fraction to get an integer" - Proof is below!)

**Claim:** if \( e \in \mathbb{Z} \), then \( \sqrt{e} \in \mathbb{Z} \) or \( \sqrt{e} \in \mathbb{R} - \mathbb{Q} \)

Case(1): \( \exists z \in \mathbb{Z} \) s.t. \( z^2 = e \). Then we can take \( \sqrt{e} = z \)

Case(2): \( \nexists z \in \mathbb{Z} \) s.t. \( z^2 = e \).

In this case, \( \sqrt{e} \notin \mathbb{Z} \). Since \( \sqrt{e} \in \mathbb{R} \), NOS \( \sqrt{e} \notin \mathbb{Q} \).
Suppose that \( \sqrt{e} = f/g \in \mathbb{Q}, f,g \in \mathbb{Z} \) and \( \gcd(g,f) = 1 \).

a) \( g = 1 \), Which implies that \( e = f^2 \).
   which contradicts to case(2) statement.

b) \( g \neq 1 \).
   Since \( \gcd(g,f) = 1 \), \( \gcd(g^2, f^2) = 1 \).
   Thus, \( \frac{f^2}{g^2} \notin \mathbb{Z} \).
   However, \( \frac{f^2}{g^2} = e \in \mathbb{Z} \)
   which contradicts to case (2) statement.

Therefore, we prove the second case doesn’t exist by contradiction. So \( b^2 - a^2 \) must be a perfect square.