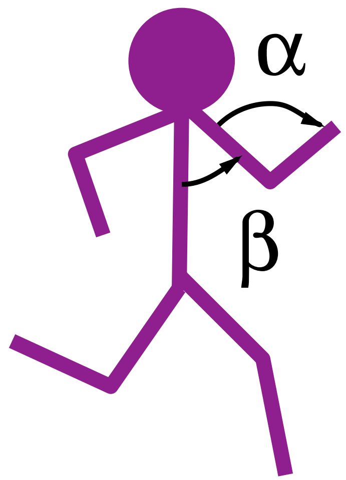
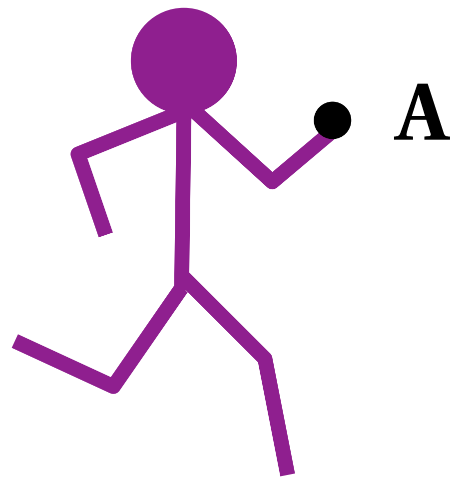
Kinematics –– the study of motion without regard to the forces that cause it.





Forward: $A = f(\alpha,\beta)$

draw graphics

Inverse: $\alpha,\beta = f^{-1}(A)$

specify fewer degrees of freedom

more intuitive control of dof
contact with the environment

calculate desired joint angles for
control

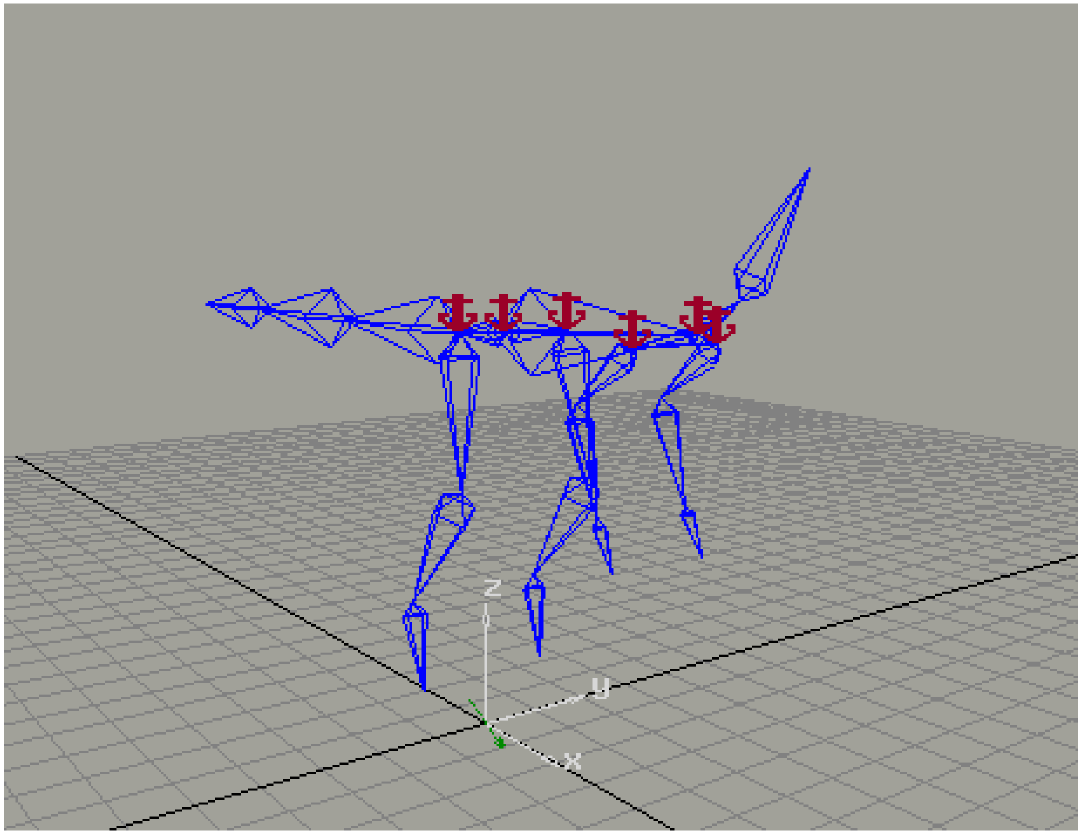
User Control of Kinematic Characters

Joint Space

- position all joints––fine level of control

Cartesian Space

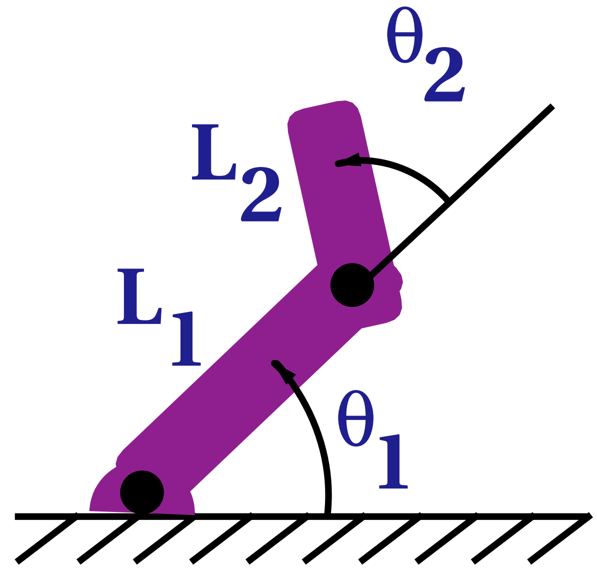
- specify environmental interactions easily
- most dof computed automatically



Forward Kinematics

$$x = L_1 \cos \theta_1 + L_2 \cos (\theta_1 + \theta_2)$$

$$y = L_1 \sin \theta_1 + L_2 \sin (\theta_1 + \theta_2)$$



$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} \text{rot } \theta_1 \end{bmatrix} \begin{bmatrix} \text{trans } L_1 \end{bmatrix} \begin{bmatrix} \text{rot } \theta_2 \end{bmatrix} \begin{bmatrix} \text{trans } L_2 \end{bmatrix}$$

Inverse Kinematics

balance –– keep center–of–mass over support polygon

control ––position vaulter's hands on line between shoulder and vault

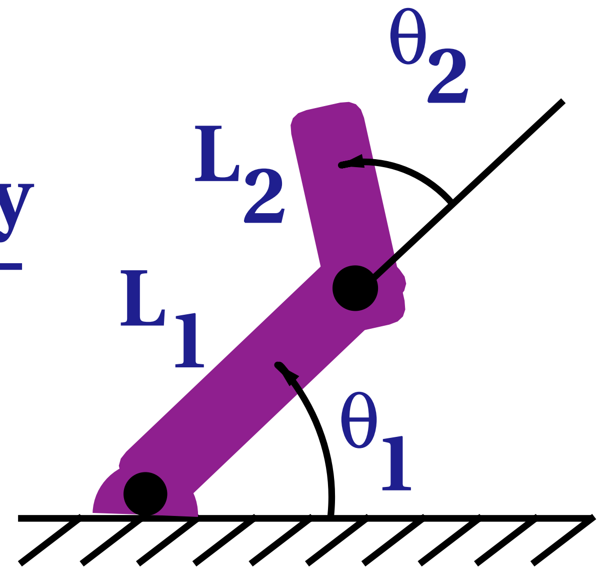
control ––compute knee angles that will give the runner the right leg length

Inverse Kinematics

$$\theta_2 = \frac{\cos\ (x^2 + y^2 - L_1^2 - L_2^2)}{2\,L_1\,L_2}$$

$$\theta_1 = \frac{-(L_2 \sin\theta_2)x + (L_1 + L_2 \cos\theta_2)y}{(L_2 \sin\theta_2)y + (L_1 + L_2 \cos\theta_2)x}$$

$$\theta = f^{-1}(x)$$



What makes IK hard? –– many dof

$$\begin{bmatrix} x_x & y_x & z_x & p_x \\ x_y & y_y & z_y & p_y \\ x_z & y_z & z_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a,b,c,d are functions of $(\theta_1, \ldots \theta_6)$

x,y,z,p are desired orientation, position of end effector

12 equations, 6 unknowns $(\theta_1, \ldots \theta_6)$

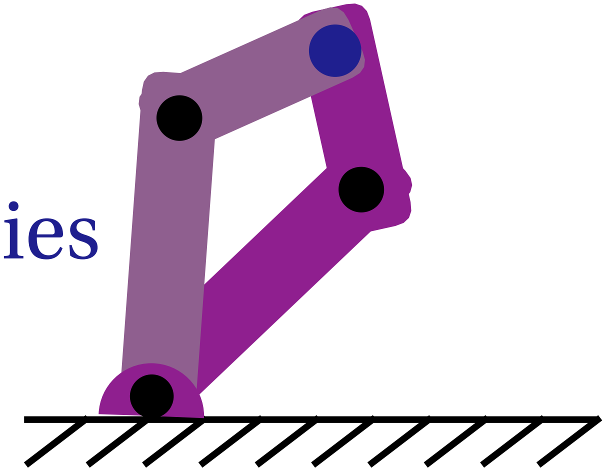
only 3 of the 9 rotation terms are independent

non–linear, transcendantal equations

What makes IK hard? -- Redundancy

a subspace $\{\theta_x\}$ defined by

$\theta(\theta_1,\ldots,\theta_n) \in \theta_x$ if $f(\theta) = X$



Add constraints to reduce redundancies

Choose solution that is
"closest" to current configuration

move outermost links the most

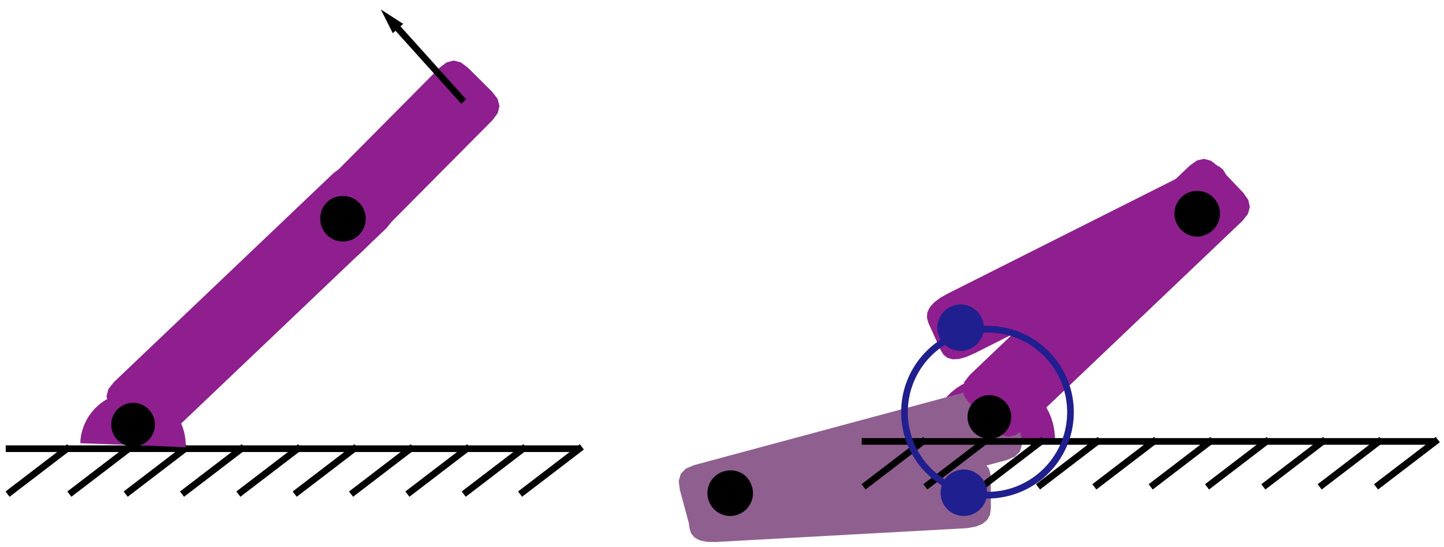
energy minimization

minimum time

What makes IK hard? -- singularities

ill–conditioned near singularities

high state space velocities for
low cartesian velocities



What makes IK hard?

- goal of "natural looking" motion
  - minimum jerk
  - equilibrium point trajectories

The Jacobian

$f(\theta) = x$ $\quad$ x is of dimension n (generally 6)

θ is of dimension m (# of dof)

Jacobian is the n x m matrix relating differential changes of θ ($d\theta$) to differential changes of x (dx)

$J(\theta)\, d\theta = dx$ where the ijth element of J is

$$J_{ij} = \frac{\delta f_i}{\delta x_j}$$

Jacobian maps velocities in state space to velocities in cartesian space

Solutions

- no solution (outside workspace, too few dof)
- multiple solutions (redundancy)
- single solution

Methods

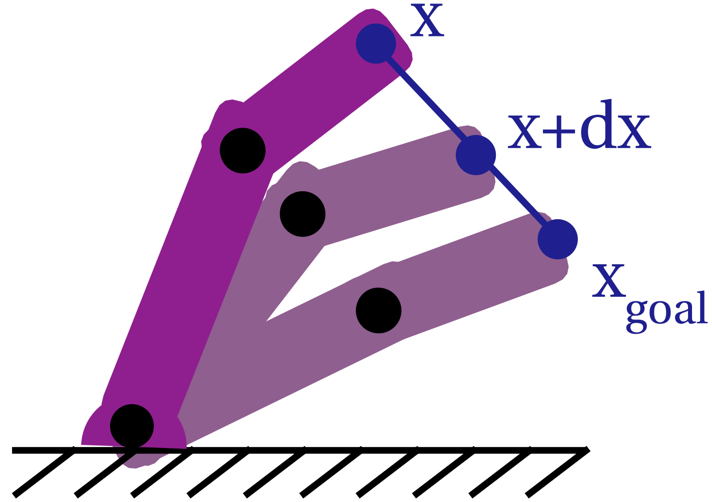
- closed form
- iterative

IK and the Jacobian

$$\theta = f^{-1}(x)$$

$$dx = J\, d\theta$$

$$d\theta = J^{-1}\, dx$$



$$\theta_{k+1} = \theta_k + \Delta t\, J^{-1}\, dx$$

linearize about θ_k

Inverting the Jacobian

J is n x m-- not square in general

compute pseudo-inverse

Singularities cause the rank of the Jacobian to change

Damped Least Squares:

find solution that minimizes

$$||J - dx||^2 + \lambda^2||d\theta||^2$$

tracking error + joint velocities

Non–linear Optimization

Zhao and Badler, TOG 1994

solution is a (local) minima of some non–linear function

objective function

constraints

non–linear optimization routine

Objective Function

position and orientation of end effector

$$P(x) = (p - x)^2$$

$$\nabla_x P(r) = 2(x-p)$$

or just position, or just orientation,
or aiming at

Formulation

minimize $G(\theta)$
subject to $a_i\theta = b_i$
$a_i\theta < b_i$

Solution

$G(\theta)$ and $\nabla G(\theta)$

use a standard numerical technique to solve