# Second Midterm

*(75 minutes open book exam)*

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Name: ____________________________
Question 1. (20 points). Suppose that $e_0, e_1, \ldots$ is a sequence of integers defined by $e_0 = 1, e_1 = 2, e_2 = 3,$
and $e_k = e_{k-1} + e_{k-2} + e_{k-3}$ for $k \geq 3$. Prove that $e_n \leq 3^n$ for all integers $n \geq 0$.

Solution. We use the strong form of Mathematical Induction.

1. Base Case. We have $e_0 \leq 3^0, e_1 \leq 3^1,$ and $e_2 \leq 3^2$.

2. Inductive Hypothesis. Suppose $e_k \leq 3^k$ for all integers $k < n$.

3. Inductive Step. We have

\[
e_n = e_{n-1} + e_{n-2} + e_{n-3} \\
\leq 3^{n-1} + 3^{n-2} + 3^{n-3} \\
\leq 3 \cdot 3^{n-1} \\
= 3^n.
\]

4. Inductive Conclusion. We have $e_n \leq 3^n$ for all non-negative integers $n$. 
Question 2. (20 points). Use a truth table to show that
$$\neg(p \lor q) \lor \neg(p \lor \neg q)$$
is equivalent to $$\neg p$$. Then, prove that this is true using De Morgan’s Law.

Solution. The truth table that shows the equivalence of
$$\neg(p \lor q) \lor \neg(p \lor \neg q)$$
and $$\neg p$$ given below.

<table>
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<tr>
<th>p</th>
<th>q</th>
<th>$$\neg(p \lor q)$$</th>
<th>$$\lor$$</th>
<th>$$\neg(p \lor \neg q)$$</th>
<th>$$\neg p$$</th>
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Table 1: Truth table.

Next we prove the equivalence using De Morgan’s Law. Writing
$$x = \neg(p \lor q) \lor \neg(p \lor \neg q)$$, we get

$$x \iff (\neg p \land \neg q) \lor (\neg p \land q)$$
$$\iff (\neg p \lor \neg p) \land (\neg q \lor q) \iff \neg p.$$  

To get the last line, we observe that $$\neg q \lor q$$ is always true and can therefore be dropped. Furthermore, for $$(\neg p \lor q) \land (\neg p \lor \neg q)$$ to be true, it must be that $$\neg p$$ is true, else one of the two terms would be false.
Question 3. (20 points). Recall Pascal’s Relation, that is, \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \). Show that
\[
\binom{2m}{m+1} + \binom{2m}{m} = \binom{2m+2}{m+1}/2,
\]
for every non-negative integer \( m \).

Solution. Set \( n = 2m \) and \( k = m + 1 \) and reorder the terms in Pascal’s Relation from back to front to get
\[
\binom{2m}{m+1} + \binom{2m}{m} = \binom{2m+1}{m+1}.
\]
The right hand side can be changed to
\[
\binom{2m+1}{m+1} = \frac{(2m+1)!}{(m+1)!m!} = \frac{(2m+2)!}{(m+1)!m!(2m+2)} = \frac{(2m+2)!}{(m+1)!(m+1)!2} = \binom{2m+2}{m+1}/2.
\]
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**Question 4.** (20 points). For all integers \( n \), let \( T(n) \) be the number of binary strings of length \( n \) that contain the substring 000. For example, for \( n = 4 \), we have the strings 1000, 0001, and 0000. All other strings of length four do not contain a substring of three consecutive zeros. Thus, \( T(4) = 3 \). Write a recurrence relation for \( T(n) \).

**Solution a.** To derive a recurrence relation, we define \( U(n) = 2^n - T(n) \), the number of binary strings of length \( n \) that do not contain 000 as a substring. To get a string of length \( n + 1 \) that contains 000, we either get the three zeros already in the first \( n \) positions or not. The former cases are counted by \( 2T(n) \), because we can add a zero or a one at the end. The latter cases that are not already counted by \( 2T(n) \) are counted by \( U(n - 3) \). To see this, take a string of length \( n - 3 \) that does not contain 000 and add 100 at the end. Only if this string ends with 100 can we add another zero to get a string that does contain 000. Hence,

\[
T(n + 1) = 2T(n) + U(n - 3) = 2T(n) - T(n - 3) + 2^{n-3}.
\]

To get this recurrence relation going, we note that \( T(n) = 0 \) for all \( n \leq 2 \) and \( T(3) = 1 \). Then \( T(4) = 2T(3) - T(0) + 1 = 3 \), which is correct.

Just out of curiosity, we compute \( T(5) = 2T(4) - T(1) + 2 = 8 \). The corresponding strings are 00000, 00001, 00010, 00011, 10000, 10001, 01000, 11000.

**Solution b.** There is another way that we can think of this recurrence. Suppose we have a binary string of length \( n + 1 \). We will let the \( i^{th} \) bit be labeled \( b_i \). We use the tree in Figure 1 to help us count the number of strings of length \( n + 1 \) that contain the substring 000. There are \( T(n) \) ways to have three consecutive zeros if \( b_{n+1} = 1 \) is one. Now, we must count the number of ways to have three consecutive zeros if \( b_{n+1} = 0 \). Well, we must consider two cases. First, if \( b_n = 1 \), then there are \( T(n - 1) \) ways to have three consecutive zeros since the three zeros must be in the first \( n - 1 \) digits. However, if the last two digits are zero, we once again must split into two cases. If \( b_{n-2} = 1 \), then there are \( T(n - 2) \) ways to obtain three consecutive zeros, but if \( b_{n-1}b_nb_{n+1} = 000 \), then all \( 2^{n-2} \) strings of this form contain three consecutive zeros. Now, we sum all of the possible (disjoint) ways to get strings containing three consecutive zeros:

\[
T(n + 1) = T(n) + T(n - 1) + T(n - 2) + 2^{n-2}
\]

Thus, we have our recurrence relation with the initial condition \( T(n) = 0 \) for all \( n \leq 2 \).

We note that the two recurrences found are in fact the same since \( T(n) = T(n - 1) + T(n - 2) + T(n - 3) + 2^{n-3} \) for all \( n \geq 0 \) implies that

\[
2T(n) - T(n - 3) + 2^{n-3} = T(n) + T(n - 1) + T(n - 2) + 2^{n-2}.
\]

The right hand side of the last equation is \( T(n + 1) \). Thus, we have \( T(n + 1) = 2T(n) - T(n - 3) + 2^{n-3} \).
Question 5. (20 points). Let $n$ be a positive integer. Prove that $\sqrt{n}$ is irrational whenever $n$ is not the square of another integer.

Solution. Let $n$ be an integer that is not the square of another integer and assume $\sqrt{n}$ is rational, that is, there are integers $i$ and $j$ such that $\sqrt{n} = \frac{i}{j}$. Then $n = \frac{i^2}{j^2}$ or, equivalently,

$$nj^2 = i^2.$$

The prime factors of $i^2$ are the prime factors of $i$ twice. It follows that the right hand side of the equation has each prime factor an even number of times. Similarly, the decomposition of $j^2$ gives each prime factor an even number of times. For the equation to hold, the decomposition of $n$ into prime factors must give each factor an even number of times. But if this is the case then $n = k^2$ for another integer $k$. Contradiction.