Fourth Homework Assignment

Write the solution to each problem on a single page. The deadline for handing in solutions is 29 March 2009.

Question 1. (20 = 10 + 10 points).

(a) Let \( n \) lines be drawn on the plane at random. We will assume that no two lines are parallel and no three lines intersect at the same point. Into how many regions do these lines divide the plane?

(b) (Problem 4.1-11 in our textbook). Find the error in the following proof that all positive integers \( n \) are equal. Let \( p(n) \) be the statement that all numbers in an \( n \)-element set of positive integers are equal. Then \( p(1) \) is true. Let \( n \geq 2 \) and write \( N \) for the set of \( n \) first positive integers. Let \( N' \) and \( N'' \) be the sets of \( n-1 \) first and \( n-1 \) last integers in \( N \). By \( p(n-1) \), all members of \( N' \) are equal, and all members of \( N'' \) are equal. Thus, the first \( n-1 \) elements of \( N \) are equal and the last \( n-1 \) elements of \( N \) are equal, and so all elements of \( N \) are equal. Therefore, all positive integers are equal.

Question 2. (20 points). Recall the Chinese Remainder Theorem stated for two positive, relatively prime moduli, \( m \) and \( n \), in Section 7. Assuming this theorem, prove the following generalization by induction on \( k \).

Claim. Let \( n_1, n_2, \ldots, n_k \) be positive, pairwise relative prime numbers. Then for every sequence of integers \( a_i \in \mathbb{Z}_{n_i}, 1 \leq i \leq k \), the system of \( k \) linear equations,

\[
 x \mod n_i = a_i,
\]

has a unique solution in \( \mathbb{Z}_N \), where \( N = \prod_{i=1}^{k} n_i \).

Question 3. (20 = 10 + 10 points).

(a) (Problem 4.2-13 in our textbook). Solve the recurrence \( T(n) = 2T(n-1) + 3^n \), with \( T(0) = 1 \).

(b) (Problem 4.2-17 in our textbook). Solve the recurrence \( T(n) = rT(n-1) + n \), with \( T(0) = 1 \). (Assume that \( r \neq 1 \)).

Question 4. (20 = 10 + 10 points).

(a) Write pseudo-code for MERGE-SORT. In your algorithm, make MERGE a separate function.

(b) How many comparisons are made during each MERGE? Prove your answer by induction.

Question 5. (20 = 10 + 10 points).

(a) Draw a recursion tree diagram for

\[
 T(n) = \begin{cases} 
 3T(n/3) + n & n \geq 3 \\
 1 & n < 3 
\end{cases}
\]

Assume \( n \) is a power of 3.

(b) Use the recursion tree to find the exact solution to the recurrence in (a).

Question 6. (20 = 4+4+4+4+4 points). (Problem 4.4-1 in our textbook). Use the Master Theorem to find the big \( \Theta \) bound on the solutions to the recurrences. For each, assume \( T(1) = 1 \) and \( n \) is a power of the appropriate integer.

(a) \( T(n) = 8T(\frac{n}{2}) + n \).

(b) \( T(n) = 8T(\frac{n}{2}) + n^3 \).

(c) \( T(n) = 3T(\frac{n}{2}) + n \).

(d) \( T(n) = T(\frac{n}{4}) + 1 \).

(e) \( T(n) = 3T(\frac{n}{2}) + n^2 \).