Thanks to several sources for ideas for lectures including Bruce Maggs, Steven Rudich and other resources.

Mmmm, blueberry pancakes eaten on Aug 25, 2013

About This Course

• Read the information on the Course web page
  • Calendar page – lecture notes available there
  • Recitation –
    • Must be signed up for either Friday or Monday recitation
    • First recitation is Fri., Aug 30 and Mon. Sept 2
  • Classwork – participation in class and recitation is part of your grade
  • Course support Staff – Graduate TA and UTAs
  • Ways to get help – office/consulting hours

Raise your hand if you have a question...

Discrete Math Vs Continuous Math

• Mathematics
  • Continuous – calculus – how things grow and change continuously
  • Discrete – quantities that can be broken into neat little pieces
Discrete vs. Continuous

Discrete
45
56109
68/3 = 22 r 2

Continuous
$\sqrt{2} = 1.4121356...$
$\pi = 3.1415926...$
$68/3 = 22.66666...$

$2^n$, n! “n choose k” $e^x$, $sin(x)$, $cos(x)$

Compare $e^x$ vs $2^n$

Discrete Mathematics

- Foundation of Computer Science
  - Dealing with digital logic, 0’s and 1’s
- Number Theory
  - What do multiples of 9 have in common?
    9 18 27 36 45 54 63 72 81 90 99 108 117 126 135
    144 153 162 171 180 189 198 207 216 ...

  - What do you know about 2354?

Number is divisible by 9 if its digits sum to a number divisible by 9

- What do you know about 2354?
  - Not divisible by 9
  - Digits sum to 14 which tells us that it has remainder 5 after dividing by 9
  - Also it is divisible by 11, will learn how later
Pancakes with a Problem

• The Chef at House of Pancakes (HoP) is very sloppy, the pancakes are all different sizes. The waitstaff, before delivering them to a customer, must rearrange them by size with the smallest on top.
• Can’t touch them! So rearrange by inserting the spatchula, and flipping the pancakes on and above the spatchula. Do as many times as needed.

Developing a Notation:
Turning pancakes into numbers

5

2

3

4

1
Developing A Notation: Turning pancakes into numbers

Exercise: How do we sort this stack? How many flips do we need?

4 Flips Are Sufficient
Algebraic Representation

$X =$ The smallest number of flips required to sort:

$\underline{? \leq X \leq ?}$

**Upper Bound**

**Lower Bound**

If we could do it in 3 flips
Flip 1 has to put 5 on bottom
Flip 2 must bring 4 to top (if it didn’t we’d need more than 3 flips).
5th Pancake Number

$P_5 = \text{The number of flips required to sort the worst case stack of 5 pancakes.}$

$4 \leq P_5 \leq ?$

How many different pancake stacks are there with 5 unique elements?
The 5th Pancake Number:
The MAX of the X’s

\[ P_5 = \text{MAX over } s \text{ stacks of 5 of } \text{MIN # of flips to sort } s \]

\[ \begin{array}{cccccc}
X_1 & X_2 & X_3 & \ldots & X_{119} & X_{120} \\
1 & 2 & 3 & \ldots & 119 & 120 \\
\end{array} \]

\[ \begin{array}{cccccc}
X_1 & X_2 & X_3 & \ldots & X_{119} & X_{120} \\
1 & 2 & 3 & \ldots & 119 & 120 \\
\end{array} \]

\[ P_n = \text{The number of flips required to sort a worst-case stack of } n \text{ pancakes.} \]

\[ P_n = \text{MAX over } s \in \text{stacks of } n \text{ pancakes of } \text{MIN # of flips to sort } s \]

What is \( P_n \) for small \( n \)?

\[ \text{Can you do } n = 0, 1, 2, 3? \]
What are the Initial Values Of $P_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$P_3 = 3$

- 1
- 3
- 2 requires 3 Flips, hence $P_3 = 3$.

ANY stack of 3 can be done in 3 flips.
- Get the big one to the bottom (· 2 flips).
- Use ·1 more flip to handle the top two.
- Hence, $P_3 = 3$.

$n^{th}$ Pancake Number

$P_n =$ Number of flips required to sort a worst case stack of $n$ pancakes.

? $\leq P_n \leq ?$  

*Upper Bound*  

*Lower Bound*
What are bounds on $P_n$, for $n > 3$?

**Upper Bound On $P_n$:**

*Bring To Top Method For $n$ Pancakes*

1. If $n = 1$, no work - we are done.
2. Else: flip pancake $n$ to top and then flip it to position $n$.

• Now use: Bring To Top Method For $n-1$ Pancakes

**Total Cost:** at most $2(n-1) = 2n - 2$ flips.
Better Upper Bound On $P_n$:
*Bring To Top Method For n Pancakes*

- If $n=2$, use one flip and we are done.
- Else: flip pancake $n$ to top and then flip it to position $n$.
- Now use: *Bring To Top Method For n-1 Pancakes*

Total Cost: at most $2(n-2) + 1 = 2n - 3$ flips.

Bring to top not always optimal for a particular stack

? $\leq P_n \leq 2n - 3$

What bounds can you prove on $P_n$?
• Suppose a stack $S$ contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.
• Any sequence of flips that sorts stack $S$ must involve one flip that inserts the spatula between that pair and breaks them apart.

• Suppose $n$ is even.
• Such a stack $S$ contains $n$ pairs that must be broken apart during any sequence that sorts stack $S$.

$n \leq P_n$

Detail: This construction only works when $n>2$
Suppose \( n \) is odd.
Such a stack \( S \) contains \( n \) pairs that must be broken apart during any sequence that sorts stack \( S \).

\[
\begin{align*}
1 & \\
3 & \\
5 & \\
7 & \\
\vdots & \\
n & \\
2 & \\
4 & \\
6 & \\
8 & \\
\vdots & \\
n-1 &
\end{align*}
\]

Detail: This construction only works when \( n > 3 \)

\[ n \leq P_n \leq 2n - 3 \quad \text{(for } n \geq 3) \]

So starting from ANY stack we can get to the sorted stack using no more than \( P_n \) flips.
From ANY stack to sorted stack in $\cdot P_n$.

From sorted stack to ANY stack in $\cdot P_n$?

Reverse the sequences we use to sort.

Hence,

From ANY stack to ANY stack in $\cdot 2P_n$. 
From ANY stack to ANY stack in \( 2P_n \).

Can you find a faster way than \( 2P_n \) flips to go from ANY to ANY?

From \( S \) to \( T \) in \( P_n \)

Rename the pancakes in \( T \) to be 1, 2, 3, ..., n.

\[
\begin{align*}
T &: 5, 2, 4, 3, 1 \\
T_{new} &: 1, 2, 3, 4, 5 \\
\pi(5), \pi(2), \pi(4), \pi(3), \pi(1)
\end{align*}
\]

Rewrite \( S \) using \( \pi(1), \pi(2), ..., \pi(n) \)

\[
\begin{align*}
S &: 4, 3, 5, 1, 2 \\
S_{new} &: \pi(4), \pi(3), \pi(5), \pi(1), \pi(2) \\
3, 4, 1, 5, 2
\end{align*}
\]

The sequence of steps that brings \( S_{new} \) to \( T_{new} \) (sorted stack) also brings \( S \) to \( T \)

The Known Pancake Numbers

\[
\begin{array}{c|c}
 n & P_n \\
1 & 1 \\
2 & 2 \\
3 & 4 \\
4 & 5 \\
5 & \\
6 & \\
7 & \\
8 & \\
9 & \\
10 & \\
11 & \\
12 & \\
13 & \\
\end{array}
\]
The Known Pancake Numbers

<table>
<thead>
<tr>
<th>n</th>
<th>P_n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
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<tr>
<td>4</td>
<td>4</td>
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<td>5</td>
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<td>6</td>
<td>7</td>
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<td>10</td>
<td>11</td>
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<td>11</td>
<td>13</td>
</tr>
<tr>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>13</td>
<td>15</td>
</tr>
</tbody>
</table>

P_{14} Is Unknown

• 14! Orderings of 14 pancakes.

• 14! = 87,178,291,200

Is This Really Computer Science?

Permutation

Any particular ordering of all \( n \) elements of an \( n \) element set \( S \) is called a permutation on the set \( S \).

Example: \( S = \{1, 2, 3, 4, 5\} \)

Example permutation: \( 5 \ 3 \ 2 \ 4 \ 1 \)

120 possible permutations on \( S \)

Each different stack of \( n \) pancakes is one of the permutations on \([1..n]\).

Representing A Permutation

- We have many choices of how to specify a permutation on \( S \). Here are two methods:

  1) List a sequence of all elements of \([1..n]\), each one written exactly once.

     Ex: \( 6 \ 4 \ 5 \ 2 \ 1 \ 3 \)

  2) Give a function \( \pi \) on \( S \) s.t. \( \pi(1) \ \pi(2) \ \pi(3) \ldots \ \pi(n) \) is a sequence listing \([1..n]\), each one exactly once.

     - Ex: \( \pi(6)=3 \ \pi(4)=2 \ \pi(5)=1 \ \pi(2)=4 \ \pi(1)=6 \ \pi(3)=5 \)
A Permutation is a NOUN

• An ordering $S$ of a stack of pancakes is a permutation.

Permute A Permutation.

I start with a permutation $S$ of pancakes. I continue to use a flip operation to permute my current permutation, so as to obtain the sorted permutation.

A Permutation is a NOUN
Permute is also a VERB

An ordering $S$ of a stack of pancakes is a permutation.

We can permute $S$ to obtain a new stack $S'$.

*Permute* also means to rearrange so as to obtain a permutation of the original.

Ultra-Useful Fact

• There are $n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$ permutations on $n$ elements.

• Proof: later
Pancake Network

• This network has n! nodes

• Assign each node the name of one of the possible n! stacks of pancakes.

• Put a wire between two nodes if they are one flip apart.

Network For n=3

Network For n=4

Pancake Network: Routing Delay

• What is the maximum distance between two nodes in the pancake network?
Pancake Network: Routing Delay

• What is the maximum distance between two nodes in the pancake network?

Pancake Network: Reliability

• If up to \( n-2 \) nodes get hit by lightning the network remains connected, even though each node is connected to only \( n-1 \) other nodes.

The Pancake Network is optimally reliable for its number of edges and nodes.

Mutation Distance

**Head Cabbage**
(Brassica oleracea capitata)

**Turnip**
(Brassica rapa)

Combinatorial “puzzle” to find the shortest series of reversals to transform one genome into another

Transforming Cabbage into Turnip: Polynomial Algorithm for Sorting Signed Permutations by Reversals

SRIDHAR HANNENHALLI
Bioinformatics, SmithKline Beecham Pharmaceuticals, King of Prussia, Pennsylvania

AND

PAVEL A. PEVZNER
University of Southern California, Los Angeles, California

Abstract. Genomes frequently evolve by reversals \( \rho(i,j) \) that transform a gene order \( \pi_1 \cdots \pi_n \) into \( \pi_1 \cdots \pi_{j-1} \pi_{j+1} \cdots \pi_n \). Reversal distance between


Over 350 citations!
Related problem – flip segments of DNA

- Solves a variant of the pancake problem
- See animation on course web site

One “Simple” Problem

A host of problems and applications at the frontiers of science

Study Bee

- Definitions
  - nth pancake number
  - Lower bound
  - Upper bound
  - Permutation

- Proof of
  - ANY to ANY in $P_n$