Announcements

- Read for next time Chap. 2.3-2.6
- Homework 2 due Tuesday
- Recitation 3 on Friday and Monday
- Microsoft Hackathon Fri 5pm for 24 hours
  - Microsoft will be holding a 24 hour hackathon called CampAppHack starting at *5pm on 9/13*. The hackathon will be focused on Windows 8 and Windows Phone, and will begin with crash courses to teach you how to develop apps for each of these platforms. After 24 hours of app hacking a panel of expert judges will choose the three top apps and award a number of awesome prizes, including an *all-expenses paid trip to Microsoft headquarters* in Redmond, WA, Microsoft Surfaces and Phones, and cash. In addition, they’ll be giving away SWAG to lucky attendees every hour, and dinner, breakfast and lunch will be provided!

Introduction to Sets

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A set is an unordered collection of objects.
  - the students in this class
  - the attendees at a Duke Football Game
- The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
- The notation \( a \in A \) denotes that \( a \) is an element of the set \( A \).
- If \( a \) is not a member of \( A \), write \( a \notin A \).
Sets – Programming Class (PC) vs. Discrete Math Class (DM)

- D.M. – formal notation, proofs
- P. C. - implementation

Describing a Set: Roster Method

- \( S = \{a, b, c, d\} \)
- Order not important
  \( S = \{a, b, c, d\} = \{b, c, a, d\} \)
- Each distinct object is either a member or not; listing more than once does not change the set.
  \( S = \{a, b, c, d\} = \{a, b, c, b, c, d\} \)
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.
  \( S = \{a, b, c, d, \ldots, z\} \)

Roster Method

- Set of all vowels in the English alphabet:
  \( V = \{a, e, i, o, u\} \)
- Set of all odd positive integers less than 10:
  \( O = \{1, 3, 5, 7, 9\} \)
- Set of all positive integers less than 100:
  \( S = \{1, 2, 3, \ldots, 99\} \)
- Set of all integers less than 0:
  \( S = \{\ldots, -3, -2, -1\} \)

Some Important Sets

- \( N = \text{natural numbers} = \{0, 1, 2, 3, \ldots\} \)
- \( Z = \text{integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \)
- \( Z^+ = \text{positive integers} = \{1, 2, 3, \ldots\} \)
- \( R = \text{set of real numbers} \)
- \( R^+ = \text{set of positive real numbers} \)
- \( C = \text{set of complex numbers} \)
- \( Q = \text{set of rational numbers} \)
Set-Builder Notation

- Specify the property or properties that all members must satisfy:
  \[ S = \{ x \mid x \text{ is a positive integer less than 100} \} \]
  \[ O = \{ x \mid x \text{ is an odd positive integer less than 10} \} \]
  \[ O = \{ x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10 \} \]
- A predicate may be used:
  \[ S = \{ x \mid P(x) \} \]
- Example: \[ S = \{ x \mid \text{Prime}(x) \} \]
- Positive rational numbers:
  \[ \mathbb{Q}^+ = \{ x \in \mathbb{R} \mid x = \frac{p}{q}, \text{ for some positive integers } p, q \} \]
- This is more precise than ...

Interval Notation

- \([a, b] = \{ x \mid a \leq x \leq b \} \]
- \([a, b) = \{ x \mid a \leq x < b \} \]
- \((a, b] = \{ x \mid a < x \leq b \} \]
- \((a, b) = \{ x \mid a < x < b \} \]
- closed interval \([a, b] \]
- open interval \((a, b) \]

Universal Set and Empty Set

- The universal set \( U \) is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized \( \emptyset \), but \( \{ \} \) also used.

Russell’s Paradox

- Let \( S \) be the set of all sets which are not members of themselves. A paradox results from trying to answer the question “Is \( S \) a member of itself?”
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question “Does Henry shave himself?”

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner
Some things to remember

- Sets can be elements of sets. How many elements?
  \{\{1,2,3\}, a, \{b,c\}\}
  \{N, Z, Q, R\}
- Are these the same?
  \emptyset \quad \{ \emptyset \}

Set Equality

**Definition**: Two sets are *equal* if and only if they have the same elements.
- Therefore if \(A\) and \(B\) are sets, then \(A\) and \(B\) are equal if and only if
- We write \(A = B\) if \(A\) and \(B\) are equal sets.
- Are these sets equal?
  \{1,3,5\} = \{3, 5, 1\}
  \{1,5,5,5,3,3,1\} = \{1,3,5\}
  \{1,2, 3,5\} = \{1, 3, 3, 5\}

Subsets

**Definition**: The set \(A\) is a *subset* of \(B\), if and only if every element of \(A\) is also an element of \(B\).
- The notation \(A \subseteq B\) is used to indicate that \(A\) is a subset of the set \(B\).
- \(A \subseteq B\) holds if and only if \(A\) is true.
- Is \(\emptyset \subseteq S\) for any set \(S\)?
- Is \(S \subseteq S\) for any set \(S\)?

Showing a Set is or is not a Subset of Another Set

- **Showing that \(A\) is a Subset of \(B\)**: To show that \(A \subseteq B\), show that if \(x\) belongs to \(A\), then \(x\) also belongs to \(B\).
- **Showing that \(A\) is not a Subset of \(B\)**: To show that \(A\) is not a subset of \(B\), \(A \not\subseteq B\), find an element \(x \in A\) with \(x \notin B\). (Such an \(x\) is a counterexample to the claim that \(x \in A\) implies \(x \in B\).)

**Examples**:
1. Is the set of all computer science majors at Duke a subset of all students at Duke?
2. Is the set of integers with squares less than 100 a subset of the set of nonnegative integers?
3. Is the set of Duke fans a subset of all students at Duke?
Another look at Equality of Sets

- Recall that two sets $A$ and $B$ are equal, denoted by $A = B$, iff
  \[ \forall x (x \in A \iff x \in B) \]
- Using logical equivalences we have that $A = B$ iff
  \[ \forall x [(x \in A \implies x \in B) \land (x \in B \implies x \in A)] \]
- This is equivalent to

Proper Subsets

**Definition:** If $A \subseteq B$, but $A \neq B$, then we say $A$ is a proper subset of $B$, denoted by $A \subset B$. If $A \subset B$, then

\[ \text{is true.} \]

Venn Diagram

Set Cardinality

**Definition:** If there are exactly $n$ distinct elements in $S$ where $n$ is a nonnegative integer, we say that $S$ is finite. Otherwise it is infinite.

**Definition:** The cardinality of a finite set $A$, denoted by $|A|$, is the number of (distinct) elements of $A$.

**Examples:**
1. $|\emptyset| = 0$
2. Let $S$ be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is $\mathbb{Z}$.

Power Sets

**Definition:** The set of all subsets of a set $A$, denoted $\mathcal{P}(A)$, is called the power set of $A$.

**Example:** If $A = \{a,b\}$ then

\[ \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \]

What is the size of $\mathcal{P}\{4, 6, 9, 12, 15\}$?

If a set has $n$ elements, then the cardinality of the power set is $2^n$. (In Chapters 5 and 6, we will discuss different ways to show this.)
Tuples

- The ordered n-tuple \((a_1,a_2,\ldots,a_n)\) is the ordered collection that has \(a_1\) as its first element and \(a_2\) as its second element and so on until \(a_n\) as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called ordered pairs.
- The ordered pairs \((a,b)\) and \((c,d)\) are equal if and only if \(a = c\) and \(b = d\).

Cartesian Product

**Definition:** The Cartesian Product of two sets \(A\) and \(B\), denoted by \(A \times B\) is the set of ordered pairs \((a,b)\) where \(a \in A\) and \(b \in B\).

**Example:**
\[ A = \{a,b\} \quad B = \{1,2,3\} \]
\[ A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\} \]

- **Definition:** A subset \(R\) of the Cartesian product \(A \times B\) is called a relation from the set \(A\) to the set \(B\). (Relations will be covered in depth in Chapter 9.)


Truth Sets of Quantifiers

- Given a predicate \(P\) and a domain \(D\), we define the truth set of \(P\) to be the set of elements in \(D\) for which \(P(x)\) is true. The truth set of \(P(x)\) is denoted by
\[ \{x \in D | P(x)\} \]
- **Example:** The truth set of \(P(x)\) where the domain is the integers and \(P(x)\) is \(\{|x| = 1\}\) is
**Boolean Algebra**

- Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*.
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set $U$. All sets are assumed to be subsets of $U$.
- Boolean algebra is fundamental to the development of Computer Science and digital logic!

**Union**

- **Definition:** Let $A$ and $B$ be sets. The *union* of the sets $A$ and $B$, denoted by $A \cup B$, is the set:  
  $$\{x | x \in A \lor x \in B\}$$
- **Example:** What is $\{1, 2, 3\} \cup \{3, 4, 5\}$?

**Intersection**

- **Definition:** The *intersection* of sets $A$ and $B$, denoted by $A \cap B$, is  
  $$\{x | x \in A \land x \in B\}$$
- Note if the intersection is empty, then $A$ and $B$ are said to be *disjoint*.
- **Example:** What is $\{1, 2, 3\} \cap \{3, 4, 5\}$?
- **Example:** What is $\{1, 2, 3\} \cap \{4, 5, 6\}$?

**Complement**

- **Definition:** If $A$ is a set, then the complement of $A$ (with respect to $U$), denoted by $\bar{A}$ is the set $U - A$
  $$\bar{A} = \{x \in U | x \notin A\}$$
  (The complement of $A$ is sometimes denoted by $A^c$.)
- **Example:** If $U$ is the positive integers less than 100, what is the complement of $\{x | x > 70\}$

**Venn Diagrams**

- Union Venn Diagram for $A \cup B$
- Intersection Venn Diagram for $A \cap B$
- Complement Venn Diagram for $\bar{A}$
Difference

• **Definition:** Let \( A \) and \( B \) be sets. The *difference* of \( A \) and \( B \), denoted by \( A - B \), is the set containing the elements of \( A \) that are not in \( B \). The difference of \( A \) and \( B \) is also called the complement of \( B \) with respect to \( A \).

\[
A - B = \{ x \mid x \in A \land x \notin B \} = A \cap \overline{B}
\]

**Venn Diagram for** \( A - B \)

The Cardinality of the Union of Two Sets

**Inclusion-Exclusion**

\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

**Example:** Let \( A \) be the math majors in your class and \( B \) be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Review Questions

**Example:** \( U = \{0,1,2,3,4,5,6,7,8,9,10\} \)
\( A = \{1,2,3,4,5\}, \)
\( B = \{4,5,6,7,8\} \)

1. \( A \cup B \)
2. \( A \cap B \)
3. \( \overline{A} \)
4. \( B \)
5. \( A - B \)
6. \( B - A \)

Symmetric Difference

**Definition:** The *symmetric difference* of \( A \) and \( B \), denoted by \( A \oplus B \) is the set

\[
(A - B) \cup (B - A)
\]

**Example:**
\[
U = \{0,1,2,3,4,5,6,7,8,9,10\}
\]
\( A = \{1,2,3,4,5\} \)
\( B = \{4,5,6,7,8\} \)

What is:

**Venn Diagram**
Set Identities

- Identity laws
  \[ A \cup \emptyset = A \quad A \cap U = A \]
- Domination laws
  \[ A \cup U = U \quad A \cap \emptyset = \emptyset \]
- Idempotent laws
  \[ A \cup A = A \quad A \cap A = A \]
- Complementation law
  \[ \overline{A} = A \]

Set Identities

- Commutative laws
  \[ A \cup B = B \cup A \quad A \cap B = B \cap A \]
- Associative laws
  \[ A \cup (B \cup C) = (A \cup B) \cup C \quad A \cap (B \cap C) = (A \cap B) \cap C \]
- Distributive laws
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Set Identities

- De Morgan’s laws
  \[ \overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B} \]
- Absorption laws
  \[ A \cup (A \cap B) = A \quad A \cap (A \cup B) = A \]
- Complement laws
  \[ A \cup \overline{A} = U \quad A \cap \overline{A} = \emptyset \]

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.
Example: Prove that $A \cap B = A \cup B$

Solution: We prove this identity by showing that:

1) $A \cap B \subseteq A \cup B$

2) $A \cup B \subseteq A \cap B$

Continued on next slide

These steps show that: $A \cup B \subseteq A \cap B$

Set-Builder Notation: Second De Morgan Law
Membership Table

Example: Construct a membership table to show that the distributive law holds.

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

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<th>B \cap C</th>
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Solution

Generalized Unions and Intersections

- Let \( A_1, A_2, \ldots, A_n \) be an indexed collection of sets. We define:

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n
\]

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n
\]

These are well defined, since union and intersection are associative.

- For \( i = 1,2,\ldots \), let \( A_i = \{i, i + 1, i + 2, \ldots\} \). Then,

\[
\bigcup_{i} A_i = \bigcup \{i, i + 1, i + 2, \ldots\} = \{1, 2, 3, \ldots\}
\]

\[
\bigcap_{i} A_i = \bigcap \{i, i + 1, i + 2, \ldots\} = \{n, n + 1, n + 2, \ldots\} = A_n
\]

Problems

- Can you conclude that \( A = B \), if \( A, B, \) and \( C \) are sets such that

\[ A \cup B = B \cup C \]

\[ A \cap B = B \cap C \]

Problems

- Can you conclude that \( A = B \), if \( A, B \) and \( C \) are sets such that

\[ A \cup B = B \cup C \quad \text{and} \quad A \cap B = B \cap C \]
Problems

• Can you conclude that \( A = B \), if \( A \), \( B \) and \( C \) are sets such that \( A \cup C = B \cup C \) and \( A \cap C = B \cap C \)?